# How (not) to do the Cholesky Decomposition: Or, how does the UK economy respond to international shocks?

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# 1 Why do we care?

- Structural models essential to what we do

- Structural relative to an intervention (some elements of model change due to policy shock or change)

- Rigobon and Sack (2004), Christiano et. al. (2007), Rubio-Ramirez, Waggonar and Zha (2010), Inoue and Killian (2013)

- How to identify: propose strategy and inference on the structural ordering of variables

- Why recursive structure?
  - Traditional, plus parts of sign-restricted and non-recursive SVARs are also often recursive
  - Example of a structural FAVAR model later

- To preview: our analysis applied to FAVAR as in Mumtaz & Surico (2009) does not appear to support structural assumptions of their models

## 1.1 SVAR identification

- Three different structures: recursive, non-recursive and sign restrictions

- Our identification strategy puts emphasis on relative variation of variables in the SVAR specification

- Can imply relative causal ordering restricitons to be verified from the data

- Somewhat related to Rigobon (2003) who relies on the change of covariances of variables at times when the variance of the policy shock increases

- SVAR has more parameters than the reduced form and solving the problem essential: reduced form characterises the probablity model fully, but how to justify restrictions

## 1.2 FAVAR Mumtaz and Surico (2009 JMCB)

- Based on Bernanke et al. (2005) and Boivin and Giannoni (2009)
  - Data-rich FAVAR: modelling interaction between the UK economy and the rest of the world
  - Large panel of around 400 international macroeconomic variables covering 17 industrialised economies
  - Plus, about 200 UK domestic economic variables covering asset prices, commodity prices, liquidity and interest rates
  - Aggregate the 600 variables into a small number of unobserved factors, and build a small-scale SVAR model based on these factors

– Plus the domestic policy rate, $R_t$, the only observable "factor" in the model

– Then, use the FAVAR to estimate the dynamic responses of a large number of home variables to foreign shocks.

- Arrangement of the "factors"
  - $F_t = \left[F_t^* : F_t^{uk}\right]$, where asterisks denotes foreign economies.
  - Model dynamics

$$\begin{bmatrix} F_t \\ R_t \end{bmatrix} = B(L) \begin{bmatrix} F_{t-1} \\ R_{t-1} \end{bmatrix} + u_t, \tag{1}$$

    where $B(L)$ is a conformable lag polynomial.
  - Unobserved factors extracted by a large panel of indicators, $X_t$, which are related to the factors by an observation equation:

$$X_t = \Lambda^F F_t + \Lambda^R R_t + \upsilon_t, \tag{2}$$

    where $\Lambda^F$ and $\Lambda^R$ are matrices of factor loadings, and $\upsilon_t$ is a vector of zero mean factor model errors.

– Mumtaz and Surico (2009) small open economy extension

* Foreign block consisting of four factors: $F_t^* = \{\Delta Y_t^*, \Pi_t^*, \Delta M_t^*, R_t^*\}$,
  · where $\Delta Y_t^*$ represents an international real activity factor,
  · $\Pi_t^*$ denotes an international inflation factor,
  · $\Delta M_t^*$ is an international liquidity factor,
  · and $R_t^*$ denotes comovements in international short-term interest rates.
* Add to this a domestic block, $F_t^{UK} = \left\{F_t^{1,UK}, \ldots, F_t^{l,UK}\right\}$, extracted from the full UK data
* And finally, the domestic monetary policy instrument, $R_t$.

- SVAR representations
  - Mumtaz and Surico (2009) consider 3 alternate SVAR representations
  - **Recursive model**: causal ordering runs from $\Delta Y_t^*$ to $\Pi_t^*$, and then progressively through $\Delta M_t^*, R_t^*$, and $F_t^{UK}$, and finally to $R_t$:

$$\begin{pmatrix} u_{\Delta Y^*} \\ u_{\Pi^*} \\ u_{\Delta M^*} \\ u_{R^*} \\ u_{F^{UK}} \\ u_R \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \times & 1 & 0 & 0 & 0 & 0 \\ \times & \times & 1 & 0 & 0 & 0 \\ \times & \times & \times & 1 & 0 & 0 \\ \times & \times & \times & \times & 1 & 0 \\ \times & \times & \times & \times & \times & 1 \end{bmatrix} \begin{pmatrix} \varepsilon_{\Delta Y^*} \\ \varepsilon_{\Pi^*} \\ \varepsilon_{\Delta M^*} \\ \varepsilon_{R^*} \\ \varepsilon_{F^{UK}} \\ \varepsilon_R \end{pmatrix} \tag{3}$$

– **Nonrecursive model** (Sims and Zha, 2006):

$$\begin{pmatrix} u_{\Delta Y^*} \\ u_{\Pi^*} \\ u_{\Delta M^*} \\ u_{R^*} \\ u_{F^{UK}} \\ u_R \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \times & 1 & 0 & 0 & 0 & 0 \\ \times & \times & 1 & \times & 0 & 0 \\ 0 & 0 & \times & 1 & 0 & 0 \\ \times & \times & \times & \times & 1 & 0 \\ \times & \times & \times & \times & \times & 1 \end{bmatrix} \begin{pmatrix} \varepsilon_{\Delta Y^*} \\ \varepsilon_{\Pi^*} \\ \varepsilon_{MD^*} \\ \varepsilon_{MS^*} \\ \varepsilon_{F^{UK}} \\ \varepsilon_R \end{pmatrix} \tag{4}$$

– **Sign restrictions**:

$$\begin{pmatrix} u_{\Delta Y^*} \\ u_{\Pi^*} \\ u_{\Delta M^*} \\ u_{R^*} \\ u_{F^{UK}} \\ u_R \end{pmatrix} = \begin{bmatrix} 1 & - & - & - & 0 & 0 \\ - & 1 & + & + & 0 & 0 \\ + & \times & 1 & - & 0 & 0 \\ + & \times & + & 1 & 0 & 0 \\ \times & \times & \times & \times & 1 & 0 \\ \times & \times & \times & \times & \times & 1 \end{bmatrix} \begin{pmatrix} \varepsilon_{AD^*} \\ -\varepsilon_{AS^*} \\ \varepsilon_{MD^*} \\ \varepsilon_{MS^*} \\ \varepsilon_{F^{UK}} \\ \varepsilon_R \end{pmatrix} \tag{5}$$

- In all three SVAR models, $F_t^*$ comes first, next $F_t^{UK}$, and finally $R_t$.
  - This ordering is what we test here.
  - Preview: This ordering is not validated by the data.
  - Why? Some "factors" of the UK economy lead the world economy
  - Which factors? UK financial markets, particularly exchange rates (Preliminary)

# 2 How is it traditionally done?

## 2.1 Permutations and Cholesky

- An illustrative example: Diebold and Yilmaz (2009 EJ)
  - Measuring spillovers in stock market volatilities across 19 countries

- Consider the reduced form VAR representation

$$x_t = \Phi x_{t-1} + \varepsilon_t$$

- By covariance stationarity, the moving average representation of the VAR exists and is given by

$$\begin{aligned} x_t &= \Theta(L)\varepsilon_t = A(L)u_t \\ \Theta(L) &= (I - \Phi L)^{-1}; A(L) = \Theta(L)Q_t^{-1}, \end{aligned}$$

where $E(u_t u_t') = I$ and $Q_t^{-1}$ is the "unique" lower-triangular Cholesky factor of the covariance matrix of $\varepsilon_t$.

- This justifies interpreting $u_t$ as the underlying structural shocks
  - Then one can potentially go ahead with constructing an index of spillovers
  - Or, for that matter, structural interpretation of the models

- But, not so simple! Uniqueness of $Q_t^{-1}$ depends on two things
  - An assumption that there is an underlying recursive ordering of variables
  - And the ordering in $x_t$ is the correct ordering

- Of course, in practise, one cannot ensure a correct ordering, except through theory
  - Diebold and Yilmaz consider averaging over all permutations
  - But find 19! permutations too hot to handle
  - Hence, consider a small number of (randomly chosen) permutations

- Klößner and Wagner (2013 JAppEconomet) provide an algorithm to explore all VAR orderings

## 2.2 So what?

- The above idea of Cholesky factorisation over "all" permutations is standard in the literature
  - However, misses the point that there is an underlying SVAR model with recursive structure
  - Does not emphasize (enough) why the Cholesky is useful

- We pose the question: Is the recursive ordering identified from the data?

- Somewhat related to Giacomini and Kitagawa (2015) and Stock and Watson (2015)

- And obtain the answer: Yes, a qualified yes!

# 3 Old wine in new bottle?

## 3.1 SVAR model and a representation

- Consider a SVAR($p$) model

$$A_0 y_t = a + \sum_{j=1}^{p} A_j y_{t-j} + \varepsilon_t,\ t = 1, \ldots, T, \tag{6}$$

where $y_t$ is an $k \times 1$ vector, $\varepsilon_t$ a $k \times 1$ vector white noise process, normally distributed with mean zero and variance-covariance matrix $\boldsymbol{\Sigma} = diag\left(\sigma_1^2, \ldots, \sigma_k^2\right)$ is a $k \times k$ positive definite diagonal matrix. $A_0, A_1, \ldots, A_p$ parameters are (at least partially) unknown $k \times k$ matrices, and $a$ is an unknown $k \times 1$ constant vector. The initial conditions $y_1, \ldots, y_p$ are given.

– Usually the idiosyncratic errors are considered IID standard normal, and the contemporaneous structural matrix, $A_0$, is left unconstrained; see, for example, Giacomini and Kitagawa (2015).

- The reduced form VAR representation of the model (6) is

$$y_t = b + \sum_{j=1}^{p} B_j y_{t-j} + u_t, \tag{7}$$

where $b = A_0^{-1} a$, $B_j = A_0^{-1} A_j$, for $j = 1, \ldots, p$, $u_t = A_0^{-1} \varepsilon_t$, and $E\left(u_t u_t'\right) = \Omega = A_0^{-1} \boldsymbol{\Sigma} \left(A_0^{-1}\right)'$.

- To obtain the reduced form, note we rescale the model and allow for heteroscedastic variances by setting the diagonal elements of $A_0$ to unity: write $A_0 = I_k - W$, where $I_k$ is the $k \times k$ identity matrix and $W$ is a $k \times k$ structural matrix with zero diagonal elements.

- This paper relates to the structure of $W$, and hence of $A_0$

- Under fairly general conditions, the reduced form parameters $b, B_1, \ldots, B_p$ are usually identified. Identification of the underlying structural parameters $a, A_0, A_1, \ldots, A_p$ requires assumptions on the structure of the SVAR.

- Our approach allows for a test of the choice of identification and causal ordering (to be made precise below).

## 3.2 Identification of causal ordering and recursive structure

- Consider the SVAR($p$) model (6) again but written in terms of the structural matrix $W$:

$$y_t = a + W y_t + \sum_{j=1}^{p} A_j y_{t-j} + \varepsilon_t,\ E\left(\varepsilon_t \varepsilon_t'\right) = \mathbf{\Sigma} = diag\left(\sigma_1^2, \ldots, \sigma_k^2\right),$$

where $W$ is a $k \times k$ matrix with zero diagonal elements. Then the reduced form is the following:

$$\begin{aligned} y_t &= (I_k - W)^{-1} a + \sum_{j=1}^{p} (I_k - W)^{-1} A_j y_{t-j} + u_t, \\ E\left(u_t u_t'\right) &= (I_k - W)^{-1} \mathbf{\Sigma} (I_k - W)^{-1\prime}. \end{aligned}$$

- Make a crucial assumption for our identification result:
  *Assumption 1 (Recursive Structure) There exists some permutation of the variables in* $y_t$*, say* $y_t^{[P]} = \left(y_t^{[1]}, \ldots, y_t^{[k]}\right)$*, for which the corresponding structural matrix* $W^{[P]}$ *is a lower triangular* $k \times k$ *matrix with zero diagonal elements. That is,* $W^{[P]} = \left(\left(w_{ij}^{[P]} : w_{ij}^{[P]} = 0 \text{ if } j \geq i\right)\right)_{i,j=1,\ldots,k}$.

- Interpretation and **Lemma 1**. $(I_k - W)^{-1} = I_k + \sum_{i=1}^{k-1} W^i$.
  - An aside: makes it convenient to calculate "direct and indirect effects".

## 3.3 Propositions

- **Proposition 1**. *Consider the SVAR($p$) model (6) with $k > 2$, with no lag structure ($p = 0$), homoscedasticity of the shocks ($\sigma_1^2 = \ldots = \sigma_k^2 = \sigma^2$), and where* **Assumption 1** *holds. Then the variable with the smallest variance ($y^{[1]}$) comes at the top of the causal order. Construct the partial covariance matrix of the other variables, after partialling out $y^{[1]}$. The variable with the smallest partial variance ($y^{[2]}$) occupies the second position in the causal order. This iterative procedure recovers the causal order $y_t^{[P]} = \left(y_t^{[1]}, \ldots, y_t^{[k]}\right)$ for the entire vector $y_t$.*

- Note: (a) partial covariances matrices easily estimated by OLS, and (b) identification is through relative variances
  - Reminiscent of Rubio-Ramírez et al. (2010) and Sims (2012)

- **Proposition 2**. *Consider the SVAR($p$) model (6) with any number of variables and any lag structure. The innovations are potentially heteroscedastic. We make* **Assumption 1** *(recursive structure), and further that the variables are in their correct recursive order. Denote the Cholesky decomposition of the reduced form error covariance matrix* $E\left(u_t^{[P]}\, {u^{[P]}}'_t\right) = \Omega$ *in (7) as* $\Omega = LL'$. *Then, the standard deviations of the idiosyncratic shocks constitute the diagonal elements of* $L$.

  - **Corollary 1**. *Suppose we can obtain a consistent estimator* $\widehat{\Omega}$ *(we may need further assumptions – Gaussian or moments). Then, the idiosyncratic error standard deviations are consistently estimated by the corresponding diagonal elements of* $\widehat{L}$.

- **Proposition 2** clearly emphasizes the precise role of the Cholesky decomposition and permutations.
  - With the correct ordering, the Cholesky factorisation correctly identifies the standard deviations of the idiosyncratic shocks.
  - However, the correct ordering is likely unknown *a priori*. Hence the (potential) need for permutations.

- **Proposition 3**. *Consider the SVAR($p$) model (6) with $k > 2$, with arbitrary lag structure ($p = 0$), arbitrary heteroscedasticity of the innovations, and where* **Assumption 1** *holds. Scale each variable by its standard deviation estimated using* **Proposition 2**. *That is:* $y_{[S]1t} = y_{1t}/\widehat{\sigma}_1, \ldots, y_{[S]kt} = y_{kt}/\widehat{\sigma}_k$. *Estimate the error covariance matrix from the reduced form VAR($p$) model based on the standardised variables. Then the variable with the smallest variance ($y_{[S]}^{[1]}$) comes at the top of the causal order. Construct the partial covariance matrix of the other variables, after partialling out* $y_{[S]}^{[1]}$. *The variable with the smallest partial variance ($y_{[S]}^{[2]}$) occupies the second position in the causal order. This iterative procedure recovers the causal order* $y_{[S]t}^{[P]} = \left( y_{[S]t}^{[1]}, \ldots, y_{[S]t}^{[k]} \right)$ *for the entire vector* $y_t$.

- One crucial implication of **Propositions 2 and 3** is that it allows us to restrict attention to a small set of admissible permutations. We start with a candidate permutation in **Proposition 2** and then this permutation is admissible if, and only if, it matches with the ordering recovered by **Proposition 3**.

- Then, one can check consistency of structural implications under all such admissible orderings, and if there is only one, this ordering is unique.

- One can also average over all such admissible orderings, or place a (Bayesian) prior over these depending, for example, on how closely they line up with underlying theory
  - This is clearly in line with DSGE-VAR (Del Negro and Schorfheide, 2004, 2009)

- Discussion and interpretation
  - Validation of structural assumptions underlying macroeconomic models is of considerable importance; for recent discussions, see Giacomini and Kitagawa (2015) and Stock and Watson (2015).
  - The above results provide identification of causal ordering under the assumption of recursive structure
  - Note: The ordering is scale invariant, and the standardization in **Proposition 3** is only to ensure that we are comparing "like-for-like"
  - Also useful in many SVARs, which are non-recursive or sign-identified, where part of the model is recursive (see application below)
  - Inference to be developed; for the moment, we use wild bootstrap

# 4 Application: UK Open Economy FAVAR

## 4.1 Data and First Results

- Quarterly data from 1974Q1 to 2005Q1 on about 600 series

- UK and 15 other OECD countries
  - US, CA, DE, FR, IT, BE, NL, PT, ES, FI, SE, NO, AT, AU, and JP

- Data converted to stationary series

- Factors extracted by principal components (Mumtaz and Surico, 2009)
  – Plus, common correlated effects (Pesaran, 2006) and dynamic factor models (Stock and Watson, 1989)
  – Finally, aggregate UK factors into a single dynamic factor (Mumtaz and Surico, 2009), but this is problematic!

## 4.2 Inference on structural ordering

- Start with reduced form VAR (7) with 3 lags (lag selection)

- Then use **Proposition 2** to compute idiosyncratic error standard deviations

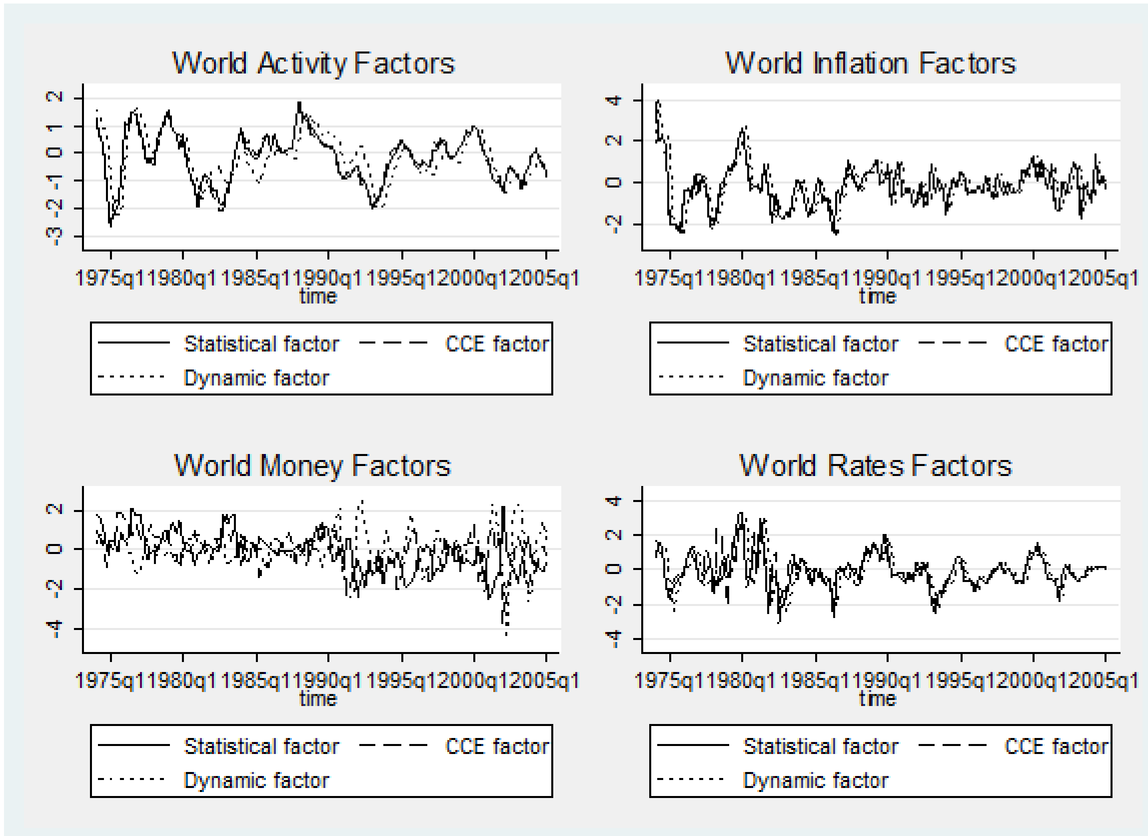


Figure 1: Extracted “Foreign” Factors

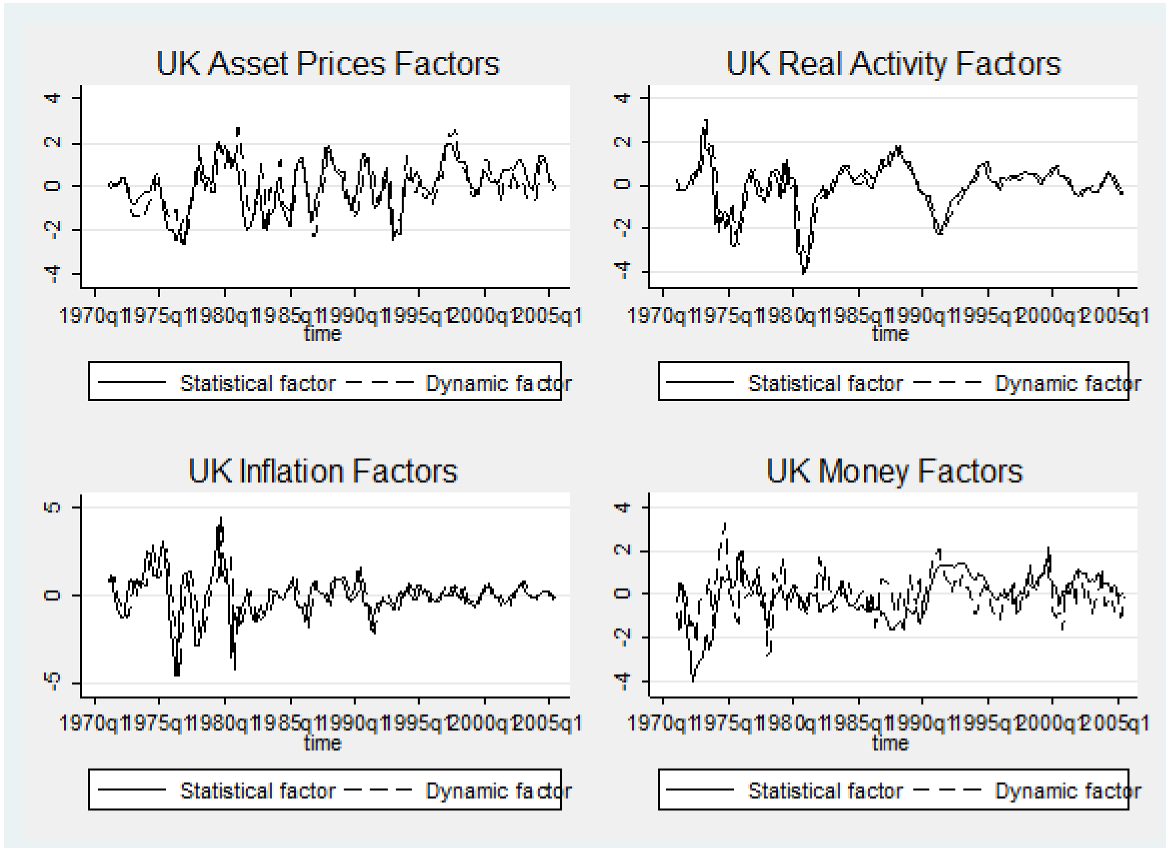


Figure 2: Extracted “Domestic” Factors

- Rescale variables ("factors") and use **Proposition 3** to infer causal order

- A unique admissible structure is supported by the data:

$$F_t^{UK} \longrightarrow \Pi_t^* \longrightarrow \Delta M_t^* \longrightarrow R_t^* \longrightarrow \Delta Y_t^* \longrightarrow R_t.$$

- "Domestic" policy rate placed unambiguously at the end of causal chain
  – Thus, monetary policy shocks can be well-identified from the SVARs.

- However, neither of the three SVAR models – recursive (3), nonrecursive (4) and sign and zero restrictions (5) – is supported by the data.

- Because the "domestic" factor, $F_t^{UK}$, comes at the top of the causal structure, which constitutes a violation of each of the above models.

- Maybe UK economy not quite like a small open economy, but rather a "medium-sized" economy
  - Some shocks from the UK economy drive the dynamics of the world economy rather than the other way round.
  - Which shocks? Now, one has to look into the black box
    * Preliminary work suggests it is the UK asset prices (exchange rates?)

# 5 Remarks and Future Extensions

- **Contemporaneous structural recursive ordering is identified from the data**

- The importance of causal ordering
  - Some structural assumptions underlying SVARs can be validated from the data
  - This paper provides identification results in this direction
  - Important implications for the way one looks at models and data, and impact of policy

- FAVAR-SVAR small open economy model for the UK
  - Monetary policy shock can be identified
  - However, features of the UK economy drive the world economy

- Potentially, asset prices – perhaps exchange rates
- Important implications for monetary policy and its transmission – impulse responses to de done
- Interesting implications for theory

- Inference needs to be developed
  - Bayesian inference, potentially leading to DSGE-SVARs.

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