Cardiff Economics Working Papers

Working Paper No. E2016/8

Skewness and Kurtosis Ratio Tests: With Applications to Multiperiod Tail Risk Analysis

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August 2016

ISSN 1749-6010

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Abstract

This article extends the variance ratio test of Lo and MacKinlay (1988) to tests of skewness and kurtosis ratios. The proposed tests are based on generalized methods of moments. In particular, overlapping observations are used and their dependencies (under the IID assumption) are explicitly modelled so that more information can be used in order to make the tests more powerful with better size properties. The proposed tests are particularly relevant to the risk management industry where risk models are estimated using daily data, although multi-period forecasts of tail risks are required for the determination of risk capital. Applications of the tests find significant higher-order nonlinear dependencies in global major equity markets. Failure to correctly model such nonlinear relationships is likely to have a negative impact on the accuracy of forecasts of multi-period tail risks.

Keywords: Skewness, kurtosis, overlapping observations, mutiperiod tail risk, Value-at-Risk

JEL Classification: C10, G11

1 Introduction

The financial crisis of 2008 has highlighted the importance of banks having sufficient capital for their trading activities. For example, if an internal risk model is used, Basel II stipulates that the market risk capital for a bank's trading portfolio should be determined by Value-at-Risk (VaR) over a 10-day horizon.[1] Since VaR is estimated using daily returns, the Basle Committee sanctions the use of a scaling law based on the variance ratio relation, i.e., the 10-day VaR is approximated as $\sqrt{10} \times$ 1-day VaR. This scaling law is clearly far from perfect since Lo and MacKinlay (1988) show that the variance ratio relationship does not hold for stock market returns.

Most risk models forecast tail risks by focusing on two main components, namely, the volatility process and the distribution function of the shocks (see for example Hsieh (1993) and Wong (2010)). While popular diagnostic tests for risk models often require that the

[1] According to Basel II, the required market risk capital on trading day t is determined as the larger of the 10-day VaR_{t-1} and the $M\times$ average of past 60 days of 10-day VaR's, where M is the multiplication factor the value of which lies between 3 and 4.

shocks and their squares are not autocorrelated, it is desirable to test for higher order dependence for two reasons. Firstly, financial returns are highly non-normal and the severity of tail risk is closely associated with higher order measures such as skewness and kurtosis. Secondly, due to the so-called intervaling effects on skewness and kurtosis (Hawawini, 1980; Lau and Wingender, 1989), the scaling law would give rise to sub-optimal multi-period forecasts of tail risks.

This article extends the variance ratio test of Lo and MacKinlay (1988) to tests of skewness and kurtosis ratios. Specifically, under the IID (independently and identically distributed) assumption, the ratios of skewness and kurtosis of single-period returns to those of h-period returns are $\sqrt{h}$ and h, respectively. Therefore, for example, if the h-period skewness is significantly more negative than $h^{-1/2}$ times the single-period skewness, multi-period forecasts of tail risk based on the IID assumption would likely to be over-optimistic.

One challenge to the proposed ratio tests is that it entails the use of higher-order statistics, which are associated with large estimation errors. In risk management, the problem is exacerbated by the requirement that tail risks be measured implicitly in a multi-period context for the purpose of risk capital determination, which results in fewer observations for risk modelling and statistical tests if non-overlapping returns are used. In order to alleviate these problems, this paper adopts the Generalized Method of Moments (GMM) used by Richardson and Smith (1991) in which overlapping observations are used and their dependencies under the IID assumption are explicitly modelled. Such an approach fully utilizes the information from the data, and thus able to provide more powerful tests. Since the higher-order ratio tests assume the existence of moments up to the eighth order, a simulation study is carried out to investigate the robustness of the proposed tests against moment condition failure. In comparison to the widely used nonlinearity test proposed by McLeod and Li (1983), the proposed ratio tests are shown to have better size properties.

In this paper, a study of multiperiod tail risks for global major equity markets is carried out. The applications of the proposed tests find signs of higher-order nonlinear dependence present in the stock returns, suggesting that the scaling-law approach to forecasting multi-period tail risks would be sub-optimal. Moreover, it is noted that in several cases the risk models studied pass the widely used Ljung-Box test of autocorrelation and the McLeod-Li nonlinearity test but fail the proposed skewness-kurtosis ratio tests.

The rest of the paper is organized as follows. Section 2 introduces some preliminary statistical properties that are useful for the derivation of the analytical results of the skewness and kurtosis ratio tests in Section 3. The next section investigates the robustness of the proposed ratio tests against moment condition failure. The empirical results of the application of the proposed tests to global equity markets are reported in Section 5. Finally, a summary is provided in Section 6.

2 Some preliminaries

2.1 Cumulants

In this paper, the analyses and results are presented in terms of cumulants. Formally, the p-th order joint cumulant of p-variate random variable $(y_1, \ldots, y_p)$, denoted as $\text{cum}(y_1, \ldots, y_p)$, is defined as the coefficient of $i^n t_1 \cdots t_p$ in the Taylor series expansion of the natural logarithm of $\text{E}\left[\exp\left(i \sum_{j=1}^{p} y_j t_j\right)\right]$. For the special case $y_j = y$, $j = 1, \ldots, p$, $\text{cum}(y_1, y_2, \ldots, y_p)$ is simply the p-th order cumulant of y. Note that $\text{cum}(y) = \text{E}(y)$ and $\text{cum}(y, y) = \text{var}(y)$.[2]

Listed below are some properties that motivate the use of cumulants in the subsequent analyses.

Lemma 1 Let z_1 and $y_1, \ldots, y_n$ be random variables whose joint cumulant exists. Then

1. $\text{cum}(y_1, \ldots, y_n)$ is symmetric in its argument.
2. $\text{cum}(y_1 + z_1, y_2, \ldots, y_n) = \text{cum}(y_1, y_2, \ldots, y_n) + \text{cum}(z_1, y_2, \ldots, y_n)$.
3. If any of $y_1, \ldots, y_n$ is independent of the remaining y's, $\text{cum}(y_1, \ldots, y_n) = 0$.
4. If a is a constant, $\text{cum}(a, y_1, \ldots, y_n) = 0$.
5. If $a_1, \ldots, a_n$ are constants, $\text{cum}(a_1 y_1, \ldots, a_n y_n) = a_1 \cdots a_n \text{cum}(y_1, \ldots, y_n)$.

2.2 Higher-order ratio relations

We shall now proceed to obtain the higher-order ratio relations based on which the proposed tests are formulated. Consider the log returns (r_t) of prices (P_t), with the former defined as $r_t = \ln(P_t/P_{t-1})$. Now define

$$\widetilde{r}_t = r_{t-h+1} + \cdots + r_t$$

as the associated h-period return at t. From now onwards, as in $\widetilde{r}_t$, we use '$\sim$' to denote that ' the variable of interest is of h-period. For simplicity, h is suppressed in all multiperiod variables in this paper. Lo and MacKinlay (1988) made use of the fact that if r_t is IID, the stock price returns should pass the variance ratio test, i.e. the relationship

$$\text{var}(\widetilde{r}_t) = h\text{var}(r_t) \tag{1}$$

[2]The appendix at the end of the paper provides further relations between higher order central moments and cumulants.

holds. The variance ratio relation can now be easily extended to higher orders in terms of cumulants, as follows. Under the IID assumption of r_t, by virtue of properties 2 and 3 of Lemma 1,

$$\widetilde{\kappa}_p = h\kappa_p, \tag{2}$$

where $\widetilde{\kappa}_p$ and κ_p are the p-th order cumulant of $\widetilde{r}_t$ and r_t respectively. The result in (2) forms the basis for the higher-order ratio tests studied in this paper. If $p = 2$, (2) reduces to (1), as the second order cumulant is simply the variance.

Since skewness and kurtosis are now widely used, it is useful to relate the result of (2) to the two statistics. Let σ^2, ρ_3 and ρ_4 be the variance, skewness and kurtosis of r_t respectively. Then under the IID assumption,

$$\widetilde{\rho}_3 = \frac{\widetilde{\kappa}_3}{\widetilde{\sigma}^3} = \frac{h}{h^{3/2}}\frac{\kappa_3}{\sigma^3} = \frac{1}{\sqrt{h}}\rho_3, \tag{3}$$

$$\widetilde{\rho}_4 = \frac{\widetilde{\kappa}_4}{\widetilde{\sigma}^4} = \frac{h}{h^2}\frac{\kappa_4}{\sigma^4} = \frac{1}{h}\rho_4. \tag{4}$$

That is, as the holding interval h increases, $\widetilde{\rho}_3$ and $\widetilde{\rho}_4$ decline at a rate of $h^{-1/2}$ and h^{-1} respectively. This is the so-called intervalling effect on skewness and kurtosis that were studied by Hawawini (1980) and Lau and Wingender (1989).

Before we proceed to derive the required tests, it is worthwhile considering the following example to illustrate why the higher order relations may not hold. Consider, for example, the two-period overlapping observations $\widetilde{r}_t = r_{t-1} + r_t$. By virtue of Lemma 1 above, the third order cumulant of $\widetilde{r}_t$ is

$$\begin{aligned} \mathrm{cum}\left(\widetilde{r}_t, \widetilde{r}_t, \widetilde{r}_t\right) &= \mathrm{cum}\left(r_{t-1}, r_{t-1}, r_{t-1}\right) + \mathrm{cum}\left(r_t, r_t, r_t\right) \\ &\quad + 3\mathrm{cum}\left(r_{t-1}, r_{t-1}, r_t\right) + 3\mathrm{cum}\left(r_{t-1}, r_t, r_t\right) \\ \widetilde{\kappa}_3 &= 2\kappa_3 + 3\mathrm{cum}\left(r_{t-1}, r_{t-1}, r_t\right) + 3\mathrm{cum}\left(r_{t-1}, r_t, r_t\right). \end{aligned} \tag{5}$$

So testing $\widetilde{\kappa}_3 = 2\kappa_3$ is equivalent to testing $\mathrm{cum}(r_{t-1}, r_{t-1}, r_t) + \mathrm{cum}(r_{t-1}, r_t, r_t) = 0$. That is, if higher order intertemporal dependence exists between r_{t-1} and r_t, skewness ratio relation does not hold.

Now, suppose r_t follows an AR(1) process:

$$r_t = m + ar_{t-1} + e_t \tag{6}$$

where m and a are constants and the innovation e_t is an IID random variable which has a finite non-zero third order cumulant or moment. Then by virtue of properties of Lemma 1,

$$\operatorname{cum}(r_{t-1}, r_{t-1}, r_t) = a \cdot \operatorname{cum}(r_{t-1}, r_{t-1}, r_{t-1}) = a \cdot \kappa_3 \neq 0. \tag{7}$$

Thus, autocorrelation in r_t would also result in the rejection of the skewness ratio relation; similar arguments also apply to the kurtosis ratio test. In short, both linear and nonlinear dependence could render the higher-order relation in (2) invalid.

3 Higher-order ratio tests

Richardson and Smith (1991) proposed a GMM approach for the variance ratio test, using (1) as a restriction in the sample moment conditions. A major contribution by Richardson and Smith is the use of analytically derived weighting matrices in the presence of overlapping returns for the GMM test. By explicitly modeling the dependencies of overlapping observations, the approach uses more information from the data and thus enjoys higher test powers and better size properties. This section extends Richardson and Smith's GMM approach to the skewness and kurtosis ratio tests.

3.1 GMM test

To apply the GMM test procedure, for each time t we construct an R-vector $f_t(r_t, \widetilde{r}_t, \theta)$ where θ is a P-vector of unknown parameters, namely μ, σ^2 and κ_j, to be determined. Each element of $f_t(\cdot)$ corresponds to a restriction, at least one of which is attributed to the higher order-ratio relation given in (2). Given the time series $\{r_t, \widetilde{r}_t\}_{t=1}^T$,

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} f_t(r_t, \widetilde{r}_t, \theta) \tag{8}$$

tends to zero as T tends to infinity if the higher order-ratio relation holds. The idea behind the GMM approach is to obtain the estimator $\widehat{\theta}$ such that it has a minimum variance-covariance matrix. Hansen (1982) showed that this can be achieved by solving the system of equations

$$D_0' S_0^{-1} g_T(\theta) = 0, \tag{9}$$

where

$$D_0 = E\left[\frac{\partial g_0(\theta)}{\partial \theta}\right], \tag{10}$$

$$S_0 = \sum_{l=-\infty}^{\infty} E\left[f_t(\cdot) f_{t-l}(\cdot)'\right]. \tag{11}$$

It can be shown that under the null hypothesis,

$$\sqrt{T}\left(\widehat{\theta} - \theta\right) \longrightarrow N\left(0, \left[D_0' S_0^{-1} D_0\right]^{-1}\right), \tag{12}$$

$$T g_T\left(\widehat{\theta}\right)' S_0^{-1} g_T\left(\widehat{\theta}\right) \longrightarrow \chi^2_{R-P}, \tag{13}$$

where $R > P$. One reason for the popularity of the GMM approach lies in its validity when D_0 and S_0 are replaced by their consistent estimators, denoted respectively as D_T and S_T. In particular, the S_T is often calculated by the two-step procedure of Hansen and Singleton (1982) or the Newey and West (1987) approach, which guarantees a positive definite weighting matrix based on sample estimates of (8).

A contribution of this article is to derive analytically, under the IID assumption, the matrix $S_0(\cdot)$ when overlapping observations are used. As Richardson and Smith (1991) have demonstrated for the variance ratio test, this approach uses more information from the data and yields desirable results such as higher test powers and better size properties. As can be seen in the following section, using an analytically derived S_0 reduces the problem to estimating only the required cumulants.

3.2 Skewness ratio test

For the skewness ratio test, f_t and D_0 are

$$f_t = \begin{bmatrix} r_t - \mu \\ (r_t - \mu)^3 - \kappa_3 \\ (\widetilde{r}_t - h\mu)^3 - h\kappa_3 \end{bmatrix}, \quad D_0 = \begin{bmatrix} -1 & 0 \\ -3\sigma^2 & -1 \\ -3h^2\sigma^2 & -h \end{bmatrix}, \tag{14}$$

with $R = 3$ and $P = 2$. Note that $\theta = (\mu \quad \kappa_3)'$. To derive the required variance-covariance matrix S_0, consider for example the covariance between the second and last elements of f_t in (14), i.e. $\operatorname{cov}((r_t - \mu)^3 - \kappa_3, (\widetilde{r}_t - h\mu)^3 - h\kappa_3)$. Since κ_3 is non-stochastic, by virtue of the properties in Lemma 1, the required covariance is simply $\text{cum}(x_t^3, \widetilde{x}_t^3)$ where

$$x_t = r_t - \mu, \tag{15}$$
$$\widetilde{x}_t = \widetilde{r}_t - h\mu. \tag{16}$$

So, the associated element of S_0 is $\sum_{l=-\infty}^{\infty} \text{cum}\left(x_t^3, \widetilde{x}_{t-l}^3\right)$, which can be denoted as $s_{1,h}^{3,3}$, where the superscripts refer to the powers of random variables and the subscripts to the periods over which the returns are measured. Using the same notation, the required covariance matrix can be written as

$$S_0 = \begin{bmatrix} s_{1,1}^{1,1} & s_{1,1}^{1,3} & s_{1,h}^{1,3} \\ s_{1,1}^{3,1} & s_{1,1}^{3,3} & s_{1,h}^{3,3} \\ s_{h,1}^{3,1} & s_{h,1}^{3,3} & s_{h,h}^{3,3} \end{bmatrix}.$$

Exploiting the overlapping dependencies and the IID assumption, the elements of S_0 are derived in the Appendix as:[3]

$$s_{1,1}^{1,1} = \sigma^2, \tag{17}$$
$$s_{1,h}^{1,3} = h\left[\kappa_4 + 3h\sigma^4\right], \tag{18}$$
$$s_{1,h}^{3,3} = h\left[\kappa_6 + (3h+12)\,\kappa_4\sigma^2 + 9\kappa_3^2 + (9h+6)\,\sigma^6\right], \tag{19}$$
$$s_{h,h}^{3,3} = h^2\kappa_6 + \left[6h^3 + 9A_h\right]\kappa_4\sigma^2 + 9A_h\kappa_3^2 + \left[9h^4 + 6B_h\right]\sigma^6, \tag{20}$$

where $A_h = h\left(2h^2+1\right)/3$ and $B_h = h^2\left(h^2+1\right)/2$. Note that if $h = 1$, $A_h = B_h = 1$ and (18) reduce to $s_{1,1}^{1,3}$, whereas both (19) and (20) simplify to $s_{1,1}^{3,3}$.

3.3 Kurtosis ratio test

For the kurtosis ratio test, the corrsponding f_t and D_0 are

$$f_t = \begin{bmatrix} r_t - \mu \\ (r_t-\mu)^2 - \sigma^2 \\ (r_t-\mu)^4 - 3\sigma^4 - \kappa_4 \\ (\widetilde{r}_t - h\mu)^4 - 3h^2\sigma^4 - h\kappa_4 \end{bmatrix}, \quad D_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -4\kappa_3 & -6\sigma^2 & -1 \\ -4h^2\kappa_3 & -6h^2\sigma^2 & -h \end{bmatrix}. \tag{21}$$

[3] The proofs in the Appendix for the analytical results of covariance matrices S_0 are made simpler using x_t and $\widetilde{x}_t$ rather than r_t and $\widetilde{r}_t$, as the former has zero mean.

Here, $R=4$, $P=3$ and $\theta = (\mu \quad \sigma^2 \quad \kappa_4)'$. Using the same notation as in the skewness ratio test, the associated weighting matrix is given by

$$S_0 = \begin{bmatrix} s_{1,1}^{1,1} & s_{1,1}^{1,2} & s_{1,1}^{1,4} & s_{1,h}^{1,4} \\ s_{1,1}^{2,1} & s_{1,1}^{2,2} & s_{1,1}^{2,4} & s_{1,h}^{2,4} \\ s_{1,1}^{4,1} & s_{1,1}^{4,2} & s_{1,1}^{4,4} & s_{1,h}^{4,4} \\ s_{h,1}^{4,1} & s_{h,1}^{4,2} & s_{h,1}^{4,4} & s_{h,h}^{4,4} \end{bmatrix},$$

where the required covariances are derived in the Appendix as

$$s_{1,1}^{1,2} = \kappa_3, \tag{22}$$

$$s_{1,1}^{2,2} = \kappa_4 + 2\sigma^4, \tag{23}$$

$$s_{1,h}^{1,4} = h\left[\kappa_5 + 10h\kappa_3\sigma^2\right], \tag{24}$$

$$s_{1,h}^{2,4} = h\left[\kappa_6 + (6h+8)\,\kappa_4\sigma^2 + (4h+6)\,\kappa_3^2 + 12h\sigma^6\right], \tag{25}$$

$$\begin{aligned} s_{1,h}^{4,4} &= h\kappa_8 + (6h+22)\,\kappa_6\sigma^2 + (4h+52)\,\kappa_5\kappa_3 + 34\kappa_4^2 \\ &\quad + (84h+120)\,\kappa_4\sigma^4 + (100h+180)\,\kappa_3^2\sigma^2 + (72h+24)\,\sigma^8, \end{aligned} \tag{26}$$

$$\begin{aligned} s_{h,h}^{4,4} &= h^2\kappa_8 + \left[12h^3 + 16A_h\right]\kappa_6\sigma^2 + \left[8h^3 + 48A_h\right]\kappa_5\kappa_3 + 34A_h\kappa_4^2 \\ &\quad + \left[36h^4 + 96hA_h + 72B_h\right]\kappa_4\sigma^4 + \left[64h^4 + 72hA_h + 144B_h\right]\kappa_3^2\sigma^2 \\ &\quad + \left[72h^2A_h + 24C_h\right]\sigma^8. \end{aligned} \tag{27}$$

In (27), $C_h = h\left(6h^4 + 10h^2 - 1\right)/15$. Similar to the case of the skewness ratio test, when $h=1$, $C_h = 1$, (24) yields $s_{1,1}^{1,4}$, (25) yields $s_{1,1}^{2,4}$ and both (26) and (27) simplify to $s_{1,1}^{4,4}$.

3.4 Joint skewness and kurtosis ratio test

We also consider a joint test based on both skewness and kurtosis relations, for the two statistics are often used together as a joint statistic (see for example the Jarque and Bera (1980) test for normality). For the case of a joint skewness and kurtosis ratio test, we have

$$f_t = \begin{bmatrix} r_t - \mu \\ (r_t-\mu)^2 - \sigma^2 \\ (r_t-\mu)^3 - \kappa_3 \\ (r_t-\mu)^4 - 3\sigma^4 - \kappa_4 \\ (\widetilde{r}_t - h\mu)^3 - h\kappa_3 \\ (\widetilde{r}_t - h\mu)^4 - 3h^2\sigma^4 - h\kappa_4 \end{bmatrix}, \quad D_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -3\sigma^2 & 0 & -1 & 0 \\ -4\kappa_3 & -6\sigma^2 & 0 & -1 \\ -3h^2\sigma^2 & 0 & -h & 0 \\ -4h^2\kappa_3 & -6h^2\sigma^2 & 0 & -h \end{bmatrix} \tag{28}$$

with $R = 6$, $P = 4$ and $\theta = (\mu \quad \sigma^2 \quad \kappa_3 \quad \kappa_4)'$. The covariance matrix is

$$S_0 = \begin{bmatrix} s_{1,1}^{1,1} & s_{1,1}^{1,2} & s_{1,1}^{1,3} & s_{1,1}^{1,4} & s_{1,h}^{1,3} & s_{1,h}^{1,4} \\ s_{1,1}^{2,1} & s_{1,1}^{2,2} & s_{1,1}^{2,3} & s_{1,1}^{2,4} & s_{1,h}^{2,3} & s_{1,h}^{2,4} \\ s_{1,1}^{3,1} & s_{1,1}^{3,2} & s_{1,1}^{3,3} & s_{1,1}^{3,4} & s_{1,h}^{3,3} & s_{1,h}^{3,4} \\ s_{1,1}^{4,1} & s_{1,1}^{4,2} & s_{1,1}^{4,3} & s_{1,1}^{4,4} & s_{1,h}^{4,3} & s_{1,h}^{4,4} \\ s_{h,1}^{3,1} & s_{h,1}^{3,2} & s_{h,1}^{3,3} & s_{h,1}^{3,4} & s_{h,h}^{3,3} & s_{h,h}^{3,4} \\ s_{h,1}^{4,1} & s_{h,1}^{4,2} & s_{h,1}^{4,3} & s_{h,1}^{4,4} & s_{h,h}^{4,3} & s_{h,h}^{4,4} \end{bmatrix}. \tag{29}$$

Most of the elements of S_0 in (29) have been provided in the above. The remaining required covariance elements are (see the Appendix for proofs)

$$\begin{aligned} s_{1,h}^{2,3} &= h\left[\kappa_5 + (3h+6)\,\kappa_3\sigma^2\right], & (30)\\ s_{1,h}^{4,3} &= h\left[\kappa_7 + (3h+18)\,\kappa_5\sigma^2 + 34\kappa_4\kappa_3 + (30h+72)\,\kappa_3\sigma^4\right], & (31)\\ s_{1,h}^{3,4} &= h\left[\kappa_7 + (6h+15)\,\kappa_5\sigma^2 + (4h+30)\,\kappa_4\kappa_3 + (66h+36)\,\kappa_3\sigma^4\right], & (32)\\ s_{h,h}^{3,4} &= h^2\kappa_7 + \left[9h^3 + 12A_h\right]\kappa_5\sigma^2 + \left[4h^3 + 30A_h\right]\kappa_4\kappa_3 \\ &\quad + \left[30h^4 + 36hA_h + 36B_h\right]\kappa_3\sigma^4. & (33) \end{aligned}$$

Again, setting $h = 1$ reduces (30) to $s_{1,1}^{2,3}$ whereas (31), (32) and (33) become $s_{1,1}^{3,4}$.

4 A simulation study of size properties

For the analytical results in the above section to hold, the moments of r_t up to eighth order need to exist (up to the sixth order for the skewness ratio test). The study by Loretan and Phillips (1994) suggests that fourth and higher order moments of a financial time series may not exist. Moreover, based on simulation experiments, de Lima (1997) finds that tests designed to have maximum power against misspecification of second or higher moments are sensitive to their nonexistence.

This section uses Monte Carlo simulations to investigate the robustness of the proposed tests when the required higher-order moments do not exist. As a benchmark for comparison, the McLeod and Li (1983) test based on squared-residual autocorrelations is also considered, for the test is not only widely used as a diagnostic check for risk models such as GARCH, but is also studied extensively for its robustness against moment condition failure in de Lima (1997).

Distributions that meet the moment condition of the proposed tests are considered first. Specifically, IID random data of sizes 250, 500 and 1000 that are distributed as standard normal and Student's t with 9 degrees of freedom are generated and subject to the skewness-kurtosis ratio tests, as well as to the McLeod and Li test.[4] Empirical test sizes are then calculated as the number of rejections of null hypothesis at a given significance level by 5,000 replications in each simulation experiment. Table 1 provides the calculated test sizes at 10%, 5% and 1% levels for h equals 5, 10 and 20 periods. McLi refers to the McLeod and Li test whereas Skew, Kurt and Joint are respectively the skewness, kurtosis and their joint ratio tests. Broadly speaking, the empirical test sizes are fairly close to their theoretical values. Closer observation finds that the higher-order ratio tests are generally under-sized at 10% and 5% levels, with noticeable improvements as the sample size increases. At a 1% level, the proposed tests tend to be mildly over-sized, with the joint skewness and kurtosis test most severe at 2.20% when $N = 500$ and $h = 20$. While the McLeod and Li test exhibits less under-sized tendencies at 10% and 5% levels, its over-sized problem at a 1% level is more severe for all values of N and h when we move from normal distribution to Student t distribution.

< Table 1 Empirical test sizes: all required moments exist >

Table 2 provides the empirical test sizes when the moment condition fails. Specifically, the IID random data are now generated from two Student's t-distributions with 3 and 5 degrees of freedom, which correspond to the existence of moments up to the second and fourth orders respectively. In the case of the skewness-kurtosis ratio tests, the under-sized tendency worsens noticeably at a 10% level and to a much smaller extent at a 5% level. Interestingly at a 1% level, the empirical test sizes remain broadly similar to those in Table 1 when the moment condition holds. For the McLeod and Li test, the over-sized problems are progressively more severe as the test size level moves from 5% to 1% and when a more heavy-tailed distribution is encountered.

< Table 2 Empirical test sizes: moment condition fails >

To sum up the above simulation study, the proposed higher order-ratio tests are robust against violations of moment conditions. At conventional significance levels, while it is under-sized at a 5% level and over-sized at a 1% level, the deviations from theoretical values are small, when compared with the widely used McLeod and Li test. The good size property and robustness against moment condition failure is important as empirical evidence suggests that higher-order moments of a financial time series may not exist.

[4] Student's t distribution with v degrees of freedom has finite moments up to the $(v-1)$-th order.

5 A study of multiperiod tail risk

This section illustrates the usefulness of higher-order ratio tests when we investigate multiperiod tail risks in global equity markets. Specifically, large and small capitalization stock index returns of four major economies are considered. Applications of the proposed tests confirm the presence of higher order dependence in these markets. No attempt is made to identify the best risk models in terms of goodness of fit, forecast, or ability to pass diagnostic tests. Only the results of a simple GARCH model with normal shocks are reported, for the aim here is to illustrate the complementary role of the proposed higher-order ratio tests. Indeed, there are many cases in which the GARCH-filtered returns (which are supposedly IID shocks) pass the McLeod and Li (1983) nonlinearity test but fail the skewness and kurtosis ratio tests. Evidence of the association of tail risks with skewness and kurtosis is also provided, implying that the forecasts of multiperiod tail risks could be sub-optimal if nonlinear dependence is not appropriately taken into account. In this sense, the information conveyed by the higher-order properties of single- and multi-period stock returns can shed light on the nature of the nonlinearities present in the financial returns.

5.1 Data and descriptive statistics

Large and small capitalization stock indexes from the US, UK, Germany and Japan are studied. Log returns are calculated and each time series of returns comprises about 2,500 observations starting from 2 January 2006 to 31 December 2015.[5] Table 3 provides the information on the number of observations, standard deviation (sd), skewness (sk) and kurtosis (ku) over the single- and multi-period horizons ($h = 5, 10$). Note that these statistics are adjusted for intervalling effects so that their expected values would remain constant for different values of h should the ratio relations hold. This is achieved by setting $sd = h^{-1/2}\widetilde{\sigma}$, $sk = h^{1/2}\widetilde{\rho}_3$ and $ku = h\widetilde{\rho}_4$.

< Table 3 Basic statistics >

It can be seen from the table that, as h increases both large- and small-capitalization stock indexes are increasingly more left-skewed and leptokurtic than would be the case if the returns were IID. To illustrate the relationship between tail risk and higer-order statistics, Figure 1 depicts, for the case of daily returns, how the value-at-risk of the eight stock indexes varies with its associated skewness and kurtosis. To control for the difference in dispersion

[5] Only trading-day returns are used in our study. While the number of observations is the same for both large and small capitalization stock indexes in each country, it varies from country to country for the period studied.

of each time series, the reported 99% VaR is obtained by dividing the first percentile of each time series of returns by the associated standard deviation. Scatter plot A and B are based on the 10-year sample studied in this article; they show that the VaR is positively correlated with skewness and negatively related with kurtosis. The relationships are much stronger in scatter plot C and D, where 30 years of data from 1986 to 2015 is used. The practical implication of these scatter plots is that a risk manager would face a higher tail risk if the return distribution is leptokurtic and skewed to the left.

< Figure 1 Skewness and VaR >

5.2 Applying the skewness-kurtosis ratio tests

The skewness-kurtosis ratio tests, McLeod and Li (1983) test, as well as the autocorrelation test by Ljung and Box (1978) are applied to the log returns. The results are reported in Table 4. In the table, LB is the Ljung-Box test, whereas McLi, Skew, Kurt and Joint refer to the same tests reported in Table 1 and 2. The number of lags used in Ljung-Box and McLeod-Li tests is 20 and 30 when h is 5 and 10 respectively. Under the null hyothesis, the reported test statistics of Skew, Kurt and Joint are distributed as Chi-square with 1, 1, and 2 degree(s) of freedom respectively. In the last three columns of the table, sd, $k3$ and $k4$ are the scaled standard deviation, standardized third and fourth order cumulant statistics for $\widetilde{\sigma}^2/\sqrt{h}$, $\widetilde{\kappa}_3/(h\sigma^{3/2})$ and $\widetilde{\kappa}_4/(h\sigma^4)$ respectively.[6] If the returns are IID, the expected values of these three statistics will not vary with h. Hence any large changes in them, especially $k3$ and $k4$, would likely be accompanied by large skewness and kurtosis test statistics.

< Table 4 Tests on raw returns >

Consistent with the literature, the squares of returns of our sample are highly autocorrelated, as is evidenced from the large McLi statistics. More importantly, the skewness and kurtosis ratio test results confirm the presence of third and fourth order depedence in the eight stock index returns; the only exception is the kurtosis ratio test of the fornightly returns of UK large stocks, which shows an insignificant test statistic of 0.9.

5.2.1 AR model

As explained in section 2, higher-order dependence could also be caused by linear autocorrelation in the daily returns. Since Table 4 shows that all except Japan large stocks fail the

[6] Note that $\widetilde{\sigma}^2$, $\widetilde{\kappa}_3$ and $\widetilde{\kappa}_4$ are estimated using the h-period returns $\widetilde{r}_t$ whereas σ is obtained from the daily returns r_t. Under the IID assumption, $k3/\sqrt{h}$ and $k4/h$ are respectively the skewness and kurtosis of h-period returns.

Ljung-Box test, an AR(5) model is used to remove linear correlation from the log returns.[7] Table 5 reports the results when the same tests are applied to the AR residuals. It can be seen that the Ljung-Box test statistics are now considerably lower in all eight cases, although linear dependence can still be detected in the residuals of US large and small stocks as well as UK small stocks.[8] Perhaps more importantly, the skewness and kurtosis ratio tests confirm the presence of nonlinear dependence in the returns.

< Table 5 Tests on AR-filtered returns >

Autocorrelation and intervalling effect The results in Table 4 and 5 reveal the impact of autocorrelation on the intervalling effect of the standardized third and fourth order cumulants. Compared with Table 4, all four large stock returns, as well as the US small stock returns in Table 5, have larger $k4$ and more negative $k3$; hence, the associated skewness and kurtosis test statistics are larger. The observed relationship is reversed for the small stock returns in the other three countries, namely UK, Germany and Japan.

At first glance, it is surprising that the AR residuals can show signs of further deviation from the null hypothesis (as indicated by higher skewness-kurtosis test statistics, as well as larger variations of $k3$ and $k4$ with respect to h). To explain such behaviour, without loss of generality, we assume here that all index returns follow an AR(1) process described by (6) and that the innovations e_t have a finite nonzero k-th order cumulant denoted as $\kappa_{e,k}$. Note that the k-th order cumulant of r_t can be wriitten as

$$\kappa_k = \left(1 - a^k\right)^{-1} \cdot \kappa_{e,k}$$

Here, we show how linear correlation could give rise to the observed behaviours of $k3$ and $k4$ in the weekly returns ($h = 5$); the same principle is also applied in the case of fortnightly returns ($h = 10$). Consider first the standardized statistic $k3$. The third order cumulant of weekly returns is

$$\text{cum}\left(\widetilde{r}_t, \widetilde{r}_t, \widetilde{r}_t\right) = \sum_{i=1}^{5} \text{cum}\left(r_i, r_i, r_i\right) + 3 \sum_{j \neq i} \text{cum}\left(r_i, r_i, r_j\right) + \sum_{i \neq j \neq k} \text{cum}\left(r_i, r_j, r_k\right) \tag{34}$$

By virtue of Lemma 1, the summand in the second term on the right of (34) is either zero, $a\kappa_3$ or $a^2\kappa_3$. Since autocorrelation in stock returns, if any, is weak, a is small and hence a^2 is negligible. Similar analyses show that $\text{cum}(r_i, r_j, r_k)$ is of even smaller value, $a^3\kappa_3$ or less. Hence, the third order cumulant of returns in Table 4 is approximately

[7] If AR(1) is used, all except Japan large- and small-cap residuals fail the Ljung-Box test.

[8] All residuals pass the Ljung-Box test when AR(15) is fitted.

$$\text{cum}\left(\widetilde{r}_t, \widetilde{r}_t, \widetilde{r}_t\right) \approx 5\kappa_3 + 12a\kappa_3. \tag{35}$$

The corresponding third order cumulants in Table 5 are

$$\text{cum}\left(\widetilde{e}_t, \widetilde{e}_t, \widetilde{e}_t\right) = 5\kappa_{e,3} = 5 \cdot \left(1 - a^3\right) \kappa_3 \approx 5\kappa_3 \tag{36}$$

since a is small. Now, declining variances (σ^2) with respect to h implies $a < 0$ for the first five index returns in Table 4. Since $\kappa_3 < 0$, (36) is less than (35), hence the AR residuals have more negative $k3$ statistics than those of the weekly returns. For the last three small stock returns, as the variance increases with h, $a > 0$ and, thus, the AR residuals give rise to less negative $k3$ statistics.

For the standardized kurtosis $k4$, we can use the same method of analysis and obtain for the weekly returns

$$\begin{aligned} \text{cum}\left(\widetilde{r}_t, \widetilde{r}_t, \widetilde{r}_t, \widetilde{r}_t\right) &\approx \sum_{i=1}^{5} \text{cum}\left(r_i, r_i, r_i, r_i\right) + 4 \sum_{i=1}^{4} \text{cum}\left(r_i, r_i, r_i, r_{i+1}\right) \\ &\approx 5\kappa_4 + 16a\kappa_4 \end{aligned} \tag{37}$$

whereas for the residuals, the cumulant is

$$\text{cum}\left(\widetilde{e}_t, \widetilde{e}_t, \widetilde{e}_t, \widetilde{e}_t\right) = 5\kappa_{e,4} = 5 \cdot \left(1 - a^4\right) \kappa_4 \approx 5\kappa_4 \tag{38}$$

Since $\kappa_4 > 0$, positive a implies that (37) is greater than (38), whereas a negative a produces the opposite outcome, which is consistent with the reported $k4$ statistics in Table 4 and 5.

5.2.2 GARCH models

To remove both linear and quadratic dependence from the index returns, AR(1)-GARCH with normal innovations is used, and Table 6 reports the test results on the filtered standardized shocks. As expected, all standardized shocks pass the McLeod-Li test. Although the AR component is of order 1, the Ljung-Box test finds significant autocorrelation only in the UK small-cap shocks when $h = 5$, thus illustrating the role of GARCH in helping to remove linear dependence from the returns.[9] Also, as a result of the GARCH filter, the magnitude of the $k3$ and $k4$ statistics, as well as the skewness-kurtosis test statistics are considerably smaller. However, significant nonlinear dependence is still present in the GARCH-filtered

[9] The readers are reminded that there are a total of six rejections of null hypothesis for the Ljung-Box test in Table 5 when AR(5) is fitted to the index returns.

returns, for none of them pass the joint skewness and kurtosis tests. Third order nonlinear dependence seems to be the main culprit, as all except UK small-cap with $h = 10$ fail the skewness ratio test. In comparison, only half of the kurtosis ratio relations fail to hold.

< Table 6 Tests on returns filtered by AR(1)-GARCH-Normal >

Stock markets are famous for their dramatic crashes during crises when markets are highly volatile. An increasingly more negative $k3$ measures as h increases implies the presence of third order nonlinear dependence in the shocks, possibly a reflection of the fact that during financial market crashes, negative returns tend to be large and persistent, resulting in much more left-skewed weekly or fortnightly returns than would otherwise be if the shocks were IID. Finally, it is remarked that varying the order of AR terms in the GARCH model has the same impact on the $k3$ and $k4$ statistics as in the case when AR(5) is fitted to the index returns.

6 Conclusion

Skewness and kurtosis ratio tests are developed using the GMM technique in this paper. In particular, overlapping observations are used in order to incorporate more information into the proposed tests. This is achieved by explicitly modelling the dependencies in the overlapping data under the IID assumption. Simulation experiments demonstrate that the proposed tests are robust to moment condition failure and exhibit good size properties in comparison to other nonlinearility tests such as the McLeod and Li test.

Applications of the skewness-kurtosis ratio tests to global major equity markets illustrate their complementary role to existing linear and nonlinear diagnostic tests. Several cases are noted in which the Ljung-Box and McLeod-Li tests fail to detect presence of dependence structures, whereas the proposed tests find violations of the skewness-kurtosis ratio relations. The ability of the tests to shed light on the nature of nonlinear dependence is particularly useful when multiperiod forecasts of tail events are required, for tail risks are found to be associated with the level of asymmetry and tail fatness of the distribution as measured by skewness and kurtosis respectively.

A Appendix

Analytical proofs for the covariance matrices S_0 used in the skewness-kurtosis ratio tests are provided here. The required covariances may be divided into three categories: covariance between products of single-period random returns (e.g. $s^{3,4}_{1,1}$), between products of single-period and h-period random returns (e.g. $s^{3,4}_{1,h}$), and between products of h-period random returns (e.g. $s^{3,4}_{h,h}$), with increasing degrees of complexity.

In all three cases, the required covariances can be obtained using the indecomposable partition method stated in Lemma 2. However, in order to facilitate an understanding (and cross verification) of the proofs, we first consider the results for the covariances between the products of single-period random returns. These are provided in A.1 where relations between cumulants and moments are introduced. A.2 provides Lemma 2, which is required for the derivation of the covariances of the products of multiperiod random variables. Examples are given to illustrate how the Lemma can be applied. A.3 derives all the required covariances involving multiperiod random returns. Finally, A.4 provides the required formulae to estimate the cumulants from the central moments in order to obtain the covariance matrix S_0 for the proposed tests.

A.1 Proofs for $S^{p,q}_{1,1}$

First consider the following formulae provided by Kendall and Stuart (1969, p.70) for expressing higher-order central moments in terms of cumulants:

$$\mu_2 = \kappa_2 = \sigma^2, \tag{39}$$

$$\mu_3 = \kappa_3, \tag{40}$$

$$\mu_4 = \kappa_4 + 3\sigma^4, \tag{41}$$

$$\mu_5 = \kappa_5 + 10\kappa_3\sigma^2, \tag{42}$$

$$\mu_6 = \kappa_6 + 15\kappa_4\sigma^2 + 10\kappa_3^2 + 15\sigma^6, \tag{43}$$

$$\mu_7 = \kappa_7 + 21\kappa_5\sigma^2 + 35\kappa_4\kappa_3 + 105\kappa_3\sigma^4, \tag{44}$$

$$\mu_8 = \kappa_8 + 28\kappa_6\sigma^2 + 56\kappa_5\kappa_3 + 35\kappa_4^2 + 210\kappa_4\sigma^4 + 280\kappa_3^2\sigma^2 + 105\sigma^8. \tag{45}$$

We shall now consider deriving an expression of $S^{p,q}_{1,1}$ $(1 \le p, q \le 4)$ in terms of cumulants using the above formulae. Under the IID assumption, x_t and x_{t-l} are independent for $l \neq 0$.[10] Thus, by virtue of Property 3 in Lemma 1, $\text{cum}(x_t^p, x_{t-l}^q) = 0$ for $l \neq 0$. Using the above moment formulae, and exploiting the fact that $\text{E}(x_t) = 0$, $s^{p,q}_{1,1} = \sum \text{cum}(x_t^p, x_{t-l}^q) = \text{cov}(x_t^p, x_t^q) = \mu_{p+q} - \mu_p\mu_q$, it is straightforward that $s^{1,1}_{1,1} = \sigma^2$ and $s^{1,2}_{1,1} = \kappa_3$. For $s^{2,2}_{1,1}$,

$$s^{2,2}_{1,1} = \text{cov}(x_t^2, x_t^2) = \mu_4 - \mu_2\mu_2.$$

[10] Readers are reminded that $x_t = r_t - \mu$; see (15).

Substituting for μ_4 using (41) and replacing μ_2 with σ^2, we have

$$s_{1,1}^{2,2} = \kappa_4 + 3\sigma^4 - \sigma^4 = \kappa_4 + 2\sigma^4.$$

Using the same principle, the other more complex covariances are derived as follows.

$$\begin{aligned}
s_{1,1}^{1,3} &= \kappa_4 + 3\sigma^4, && (46)\\
s_{1,1}^{1,4} &= \kappa_5 + 10\kappa_3\sigma^2, && (47)\\
s_{1,1}^{2,3} &= \kappa_5 + 9\kappa_3\sigma^2, && (48)\\
s_{1,1}^{2,4} &= \kappa_6 + 14\kappa_4\sigma^2 + 10\kappa_3^2 + 12\sigma^6, && (49)\\
s_{1,1}^{3,3} &= \kappa_6 + 15\kappa_4\sigma^2 + 9\kappa_3^2 + 15\sigma^6, && (50)\\
s_{1,1}^{3,4} &= s_{1,1}^{4,3} = \kappa_7 + 21\kappa_5\sigma^2 + 34\kappa_4\kappa_3 + 102\kappa_3\sigma^4 && (51)\\
s_{1,1}^{4,4} &= \kappa_8 + 28\kappa_6\sigma^2 + 56\kappa_5\kappa_3 + 34\kappa_4^2 + 204\kappa_4\sigma^4 + 280\kappa_3^2\sigma^2 + 96\sigma^8. && (52)
\end{aligned}$$

Letting $h = 1$ in, for example, (19) and (20) will give rise to the same formula for $s_{1,1}^{3,3}$ in (50) above. One important observation to be made here is that $s_{1,1}^{p,q}$ contains the basic *structure* for $s_{1,h}^{p,q}$ and $s_{h,h}^{p,q}$. Take the case of $p = q = 4$ as another example; the right hand sides of (26) and (27) in the kurtosis ratio test share the same cumulant terms with $s_{1,1}^{4,4}$ in (52): κ_8, $\kappa_6\sigma^2$, $\kappa_5\kappa_3$, κ_4^2, $\kappa_4\sigma^4$, $\kappa_3^2\sigma^2$ and σ^8. Moreover, when $h = 1$, $A_h = B_h = C_h = 1$, yielding the same coefficients for all cumulant terms in $s_{1,1}^{p,q}$, $s_{1,h}^{p,q}$ and $s_{h,h}^{p,q}$, where $1 \leq p, q \leq 4$. Therefore, as can be seen in A.3 below, h^p $(1 \leq p \leq 4)$, A_h, B_h and C_h reflect the effects of having h-period returns in place of single-period returns under the null hypothesis of independent returns.

A.2 Cumulant of products of random variables

The above shows how $s_{1,1}^{p,q}$ can be obtained using the formulae provided by Kendall and Stuart (1969). However, things become complicated when multiperiod random returns are involved. Since the required covariances are essentially cumulants of the products of random variables, we introduce here the concept of an indecomposable partition used by Brillinger (1975, Section 2.3) in order to obtain the cumulants of the products of x_t.

Definition *Consider a partition* $P_1 \cup \cdots \cup P_M$ *of the table of entries (not necessarily rectangular) given below*

$$\begin{matrix} (1,1) & \cdots & (1,J_1) \\ \vdots & & \vdots \\ (I,1) & \cdots & (1,J_I) \end{matrix}$$

Sets $P_{m'}$ and $P_{m''}$ are said to hook if there exists $(i_1, j_1) \in P_{m'}$ and $(i_2, j_2) \in P_{m''}$ such that $i_1 = i_2$; that is (i_1, j_1) and (i_2, j_2) are from the same row. $P_{m'}$ and $P_{m''}$ are said to communicate if there exists a sequence of sets $P_{m_1} = P_{m'}, P_{m_2}, \ldots, P_{m_N} = P_{m''}$ such that P_{m_n} and $P_{m_{n+1}}$ hook for $n = 1, \ldots, N-1$. A partition is said to be indecomposable if all of its sets communicate.

Each row in the table above corresponds to a product of random returns in our paper. So, $I = 2$, as we need only covariances that are second order cumulants. Take the case of $\mathrm{cum}\big(x_t^3, \widetilde{x}_{t-l}^4\big)$ in $s_{1,h}^{3,4}$ for illustration, we can let the first row of entries in the above table correspond to x_t^3, whereas the second row correspond to $\widetilde{x}_{t-l}^4$, so that $J_1 = 3$ and $J_2 = 4$. An indecomposable partition as defined above is one that contains at least a set in which at least one element is from x_t^3 and the other from $\widetilde{x}_{t-l}^4$.

The definition of an indecomposable partition is used by Brillinger (1975) to obtain the joint cumulant of products of random variables, as presented in Lemma 2 below.

Lemma 2 *Consider the (two way) I random variables*

$$Y_i = \prod_{j=1}^{J_i} X_{ij},$$

where $j = 1, \ldots, J_i$ and $i = 1, \ldots, I$. *The joint cumulant cum$(Y_1, \ldots, Y_I)$ is given by*

$$\sum_{\mathbf{P}} \mathrm{cum}\left(X_{ij}; ij \in P_1\right) \cdots \mathrm{cum}\left(X_{ij}; ij \in P_M\right)$$

where the summation is over all indecomposable partitions $P = P_1 \cup \cdots \cup P_M$.

Example 1 Consider the simple case of $\mathrm{cum}\big(x_t^2, x_{t-l}^2\big)$ in $s_{1,1}^{2,2}$. Then in the notation of Lemma 2, $Y_1 = X_{11}X_{12}$ and $Y_2 = X_{21}X_{22}$, which correspond to x_t^2 and x_{t-l}^2 respectively. Applying Lemma 1 and making use of the fact that $\mathrm{E}(x_t) = \mathrm{E}(\widetilde{x}_{t-l}) = 0$,

$$\begin{aligned} \mathrm{cum}\left(Y_1, Y_2\right) &= \mathrm{cum}\left(X_{11}, X_{12}, X_{21}, X_{22}\right) \\ &\quad + \mathrm{cum}\left(X_{11}, X_{21}\right) \mathrm{cum}\left(X_{12}, X_{22}\right) + \mathrm{cum}\left(X_{11}, X_{22}\right) \mathrm{cum}\left(X_{12}, X_{21}\right), \end{aligned}$$

which gives rise to

$$\mathrm{cum}\left(x_t^2, x_{t-l}^2\right) = \mathrm{cum}\left(x_t, x_t, x_{t-l}, x_{t-l}\right) + 2\mathrm{cum}\left(x_t, x_{t-l}\right)^2. \tag{53}$$

Note that $\mathrm{cum}(x_t, x_t)\mathrm{cum}(x_{t-l}, x_{t-l})$ is not an indecomposable partition because there is no cumulant term that links the x_t^2 and x_{t-l}^2 together.

A.3 Proofs for $S^{p,q}_{1,h}$ and $S^{p,q}_{h,h}$

Here, we introduce some preliminary results and a simplified notation, then proceed to derive the required covariances involving multiperiod random returns using Lemma 2.

A.3.1 Preliminary results

There are two properties of x_t which render the derivation of covariance matrices S_0 relatively straightforward. Firstly, $\mathrm{E}(x_t) = 0$. Secondly, x_t and x_{t-l} are independent except for $l = 0$. The first property enables us to ignore all indecomposable partitions that result in $\mathrm{E}(x_t)$ as a cumulant term. By virtue of Lemma 1, the second property implies that for j random variable x's at time t or $t-l$, we have

$$\mathrm{cum}\left(x_t, \ldots, x_{t-l}\right) = \begin{cases} \kappa_j & \text{if } l = 0, \\ 0 & \text{if } l \neq 0. \end{cases} \tag{54}$$

If the j random variables are a mixture of x_t's and h-period random returns $\widetilde{x}_{t-l}$'s,

$$\mathrm{cum}\left(x_t, \ldots, \widetilde{x}_{t-l}\right) = \begin{cases} \kappa_j & \text{for } 1-h \leq l \leq 0, \\ 0 & \text{for } l > 0. \end{cases} \tag{55}$$

Finally, for j h-period random returns $\widetilde{x}$'s at time t or $t-l$,

$$\mathrm{cum}\left(\widetilde{x}_t, \ldots, \widetilde{x}_{t-l}\right) = \begin{cases} \left(h - |l|\right)\kappa_j & \text{for } |l| < h, \\ 0 & \text{for } |l| \geq h. \end{cases} \tag{56}$$

A.3.2 Notation

To derive the required covariances of multiperiod returns, it is helpful to to simplify the notation in the following way. We denote the j-th order joint cumulant of random variables $y_1, .., y_j$ by $\langle y_1 \cdots y_j \rangle$, that is

$$\mathrm{cum}\left(y_1, .., y_j\right) = \langle y_1 \cdots y_j \rangle .$$

Suppose for example $y_1 = y_2 = u$ and $y_3 = \cdots = y_j = v$. Then the cumulant can be simply written as

$$\mathrm{cum}\left(y_1, .., y_j\right) = \left\langle u^2 v^{j-2} \right\rangle .$$

Note that $\langle \cdot \rangle$ does not represent the cumulant of the products of random variables; for instance, $\langle x^3 \rangle = \mathrm{cum}(x, x, x) \neq \mathrm{cum}(x^3)$.

A.3.3 Covariances for skewness ratio test

The covariances between single-period returns are already provided in A.1. Next, we consider covariances that involve h-period returns. First, consider the simple case of $s^{1,3}_{1,h} = \sum \text{cum}\big(x_t, \widetilde{x}^3_{t-l}\big)$. Applying Lemma 2,

$$\text{cum}\left(x_t, \widetilde{x}^3_{t-l}\right) = \left\langle x_t \widetilde{x}^3_{t-l}\right\rangle + 3\left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}^2_{t-l}\right\rangle$$

for $1-h \leq l \leq 0$, zero otherwise. Using results (55) and (56),

$$\text{cum}\left(x_t, \widetilde{x}^3_{t-l}\right) = \kappa_4 + 3h\sigma^4,$$

so $s^{1,3}_{1,h} = h\left[\kappa_4 + 3h\sigma^4\right]$. Similarly, for $1-h \leq l \leq 0$,

$$\begin{aligned} \text{cum}\left(x^3_t, \widetilde{x}^3_{t-l}\right) &= \left\langle x^3_t \widetilde{x}^3_{t-l}\right\rangle + 3\left\langle x^3_t \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}^2_{t-l}\right\rangle + 3\left\langle x_t \widetilde{x}^3_{t-l}\right\rangle \left\langle x^2_{t-l}\right\rangle \\ &\quad +9\left\langle x^2_t \widetilde{x}^2_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle + 9\left\langle x^2_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}^2_{t-l}\right\rangle \\ &\quad +9\left\langle x^2_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}^2_{t-l}\right\rangle + 6\left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \qquad (57) \\ &= \kappa_6 + (3h+12)\,\kappa_4\sigma^2 + 9\kappa^2_3 + (9h+6)\,\sigma^6, \end{aligned}$$

which if multiplied by h gives rise to $s^{3,3}_{1,h}$. To see how the number of each type of indecomposable partition is obtained in (57), take $\langle x^2_t \widetilde{x}_{t-l}\rangle \left\langle x_t \widetilde{x}^2_{t-l}\right\rangle$ as an example: there are three ways of choosing x^2_t from x^3_t and three ways of choosing $\widetilde{x}_{t-l}$ from $\widetilde{x}^3_{t-l}$ to yield $\langle x^2_t \widetilde{x}_{t-l}\rangle$; there is only one left way for the remaining random variables to form $\left\langle x_t \widetilde{x}^2_{t-l}\right\rangle$. So, the required number is $3 \times 3 \times 1 = 9$.

Now in the case of $\text{cum}\big(\widetilde{x}^3_t, \widetilde{x}^3_{t-l}\big)$ in $s^{3,3}_{h,h}$, each term in the sum of products of cumulants will retain the *same form* as the right hand side of (57), and replacing x_t with $\widetilde{x}_t$ yields the expression for $\text{cum}\big(\widetilde{x}^3_t, \widetilde{x}^3_{t-l}\big)$. So, making use of (56),

$$\begin{aligned} s^{3,3}_{h,h} &= \sum (h-|l|)\,\kappa_6 + \left[6h\sum (h-|l|) + 9\sum (h-|l|)^2\right]\kappa_4\sigma^2 \\ &\quad +9\sum (h-|l|)^2\,\kappa^2_3 + \left[9h^2\sum (h-|l|) + 6\sum (h-|l|)^3\right]\sigma^6, \end{aligned}$$

where the summation is from $l = -h+1, .., h-1$. Note that $\sum (h-|l|) = h^2$, $\sum (h-|l|)^2 = A_h$ and $\sum (h-|l|)^3 = B_h$, and this completes the proof for the expressin of $s^{3,3}_{h,h}$ in (20).

A.3.4 Covariances for kurtosis ratio test

From the above derivations of $S_{1,h}^{1,3}$ and $S_{1,h}^{3,3}$, we can see that covariances between products of single- and h-period random returns yield a simple multiple of h, and provide the basic form for more complex covariances between products of h-period random returns. These steps of proof are similar for covariances in the kurtosis ratio test. So we have

$$\begin{aligned} s_{1,h}^{1,4} &= \sum \operatorname{cum}\left(x_t, \widetilde{x}_{t-l}^4\right) \\ &= \sum \left[\left\langle x_t \widetilde{x}_{t-l}^4 \right\rangle + 4\left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}_{t-l}^3\right\rangle + 6\left\langle x_t \widetilde{x}_{t-l}^2\right\rangle \left\langle \widetilde{x}_{t-l}^2\right\rangle\right] \\ &= h\left[\kappa_5 + 10h\kappa_3\sigma^2\right]. \end{aligned}$$

Also, multiplying by h the following cumulant

$$\begin{aligned} \operatorname{cum}\left(x_t^2, \widetilde{x}_{t-l}^4\right) &= \left\langle x_t^2 \widetilde{x}_{t-l}^4\right\rangle + 6\left\langle x_t^2 \widetilde{x}_{t-l}^2\right\rangle \left\langle \widetilde{x}_{t-l}^2\right\rangle + 8\left\langle x_t \widetilde{x}_{t-l}^3\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \\ &\quad +4\left\langle x_t^2 \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}_{t-l}^3\right\rangle + 6\left\langle x_t \widetilde{x}_{t-l}^2\right\rangle \left\langle x_t \widetilde{x}_{t-l}^2\right\rangle + 12\left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}_{t-l}^2\right\rangle \\ &= \kappa_6 + (6h+8)\kappa_4\sigma^2 + (4h+6)\kappa_3^2 + 12h\sigma^6 \end{aligned}$$

yields $s_{1,h}^{2,4}$. The case for $s_{1,h}^{4,4}$ is more complex; the indecomposable partitions of $\operatorname{cum}\left(x_t^4, \widetilde{x}_{t-l}^4\right)$ are

$$\begin{aligned} &\left\langle x_t^4 \widetilde{x}_{t-l}^4\right\rangle + 6\left\langle x_t^4 \widetilde{x}_{t-l}^2\right\rangle \left\langle \widetilde{x}_{t-l}^2\right\rangle + 6\left\langle x_t^2 \widetilde{x}_{t-l}^4\right\rangle \left\langle x_t^2\right\rangle + 16\left\langle x_t^3 \widetilde{x}_{t-l}^3\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \\ &+4\left\langle x_t^4 \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}_{t-l}^3\right\rangle + 4\left\langle x_t \widetilde{x}_{t-l}^4\right\rangle \left\langle x_t^3\right\rangle + 24\left\langle x_t^3 \widetilde{x}_{t-l}^2\right\rangle \left\langle x_t \widetilde{x}_{t-l}^2\right\rangle + 24\left\langle x_t^2 \widetilde{x}_{t-l}^3\right\rangle \left\langle x_t^2 \widetilde{x}_{t-l}\right\rangle \\ &+18\left\langle x_t^2 \widetilde{x}_{t-l}^2\right\rangle \left\langle x_t^2 \widetilde{x}_{t-l}^2\right\rangle + 16\left\langle x_t^3 \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}^3\right\rangle \\ &+36\left\langle x_t^2 \widetilde{x}_{t-l}^2\right\rangle \left\langle x_t^2\right\rangle \left\langle \widetilde{x}_{t-l}^2\right\rangle + 48\left\langle x_t^3 \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}_{t-l}^2\right\rangle \\ &+48\left\langle x_t \widetilde{x}_{t-l}^3\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t^2\right\rangle + 72\left\langle x_t^2 \widetilde{x}_{t-l}^2\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \\ &+16\left\langle x_t^3\right\rangle \left\langle \widetilde{x}_{t-l}^3\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle + 24\left\langle x_t^3\right\rangle \left\langle x_t \widetilde{x}_{t-l}^2\right\rangle \left\langle \widetilde{x}_{t-l}^2\right\rangle + 24\left\langle x_t^2 \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}_{t-l}^3\right\rangle \left\langle x_t^2\right\rangle \\ &+36\left\langle x_t^2 \widetilde{x}_{t-l}\right\rangle \left\langle x_t^2 \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}_{t-l}^2\right\rangle + 36\left\langle x_t \widetilde{x}_{t-l}^2\right\rangle \left\langle x_t \widetilde{x}_{t-l}^2\right\rangle \left\langle x_t^2\right\rangle + 144\left\langle x_t^2 \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}^2\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \\ &+72\left\langle x_t^2\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle \widetilde{x}_{t-l}^2\right\rangle + 24\left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle \left\langle x_t \widetilde{x}_{t-l}\right\rangle. \end{aligned} \tag{58}$$

In the above, only $\left\langle \widetilde{x}_{t-l}^2\right\rangle$ and $\left\langle \widetilde{x}_{t-l}^3\right\rangle$ yield a factor h. Thus

$$\begin{aligned} \operatorname{cum}\left(x_t^4, \widetilde{x}_{t-l}^4\right) &= \kappa_8 + (6h+22)\kappa_6\sigma^2 + (4h+52)\kappa_5\kappa_3 + 34\kappa_4^2 \\ &\quad + (84h+120)\kappa_4\sigma^4 + (100h+180)\kappa_3^2\sigma^2 + (72h+24)\sigma^8, \end{aligned}$$

for $l = 1-h, .., 0$. Thus, multiplying the above by h yields $s^{4,4}_{1,h}$. Replacing x^4_t with $\widetilde{x}^4_t$ in (58) gives us $\text{cum}\left(\widetilde{x}^4_t, \widetilde{x}^4_{t-l}\right)$ which, after applying the result of (56), yields

$$\begin{aligned}
&(h-|l|)\,\kappa_8 + \left[12h\,(h-|l|) + 16\,(h-|l|)^2\right]\kappa_6\sigma^2 \\
&+ \left[8h\,(h-|l|) + 48\,(h-|l|)^2\right]\kappa_5\kappa_3 + 34\,(h-|l|)^2\,\kappa_4^2 \\
&+ \left[36h^2\,(h-|l|) + 96\,(h-|l|)^2 + 72\,(h-|l|)^3\right]\kappa_4\sigma^4 \\
&+ \left[64h^2\,(h-|l|) + 72h\,(h-|l|)^2 + 144\,(h-|l|)^3\right]\kappa_3^2\sigma^2 \\
&+ \left[72h^2\,(h-|l|)^2 + 24\,(h-|l|)^4\right]\sigma^8.
\end{aligned}$$

Summing the above from $l = -h+1$ to $h-1$ and noting $\sum_{l=-h+1}^{h-1}(h-|l|)^4 = C(h)$, we have the required covariance.

A.3.5 Covariances for the joint skewness and kurtosis ratio test

The remaining covariances to be derived for the joint skewness and kurtosis ratio test are $s^{2,3}_{1,h}$, $s^{4,3}_{1,h}$, $s^{3,4}_{1,h}$ and $s^{3,4}_{h,h}$. Using the same method as above,

$$\begin{aligned}
s^{2,3}_{1,h} &= \sum \text{cum}\left(x^2_t, \widetilde{x}^3_{t-l}\right) \\
&= \sum \left[\left\langle x^2_t\widetilde{x}^3_{t-l}\right\rangle + 6\left\langle x_t\widetilde{x}^2_{t-l}\right\rangle\left\langle x_t\widetilde{x}_{t-l}\right\rangle + 3\left\langle x^2_t\widetilde{x}_{t-l}\right\rangle\left\langle \widetilde{x}^2_{t-l}\right\rangle\right] \\
&= h\left[\kappa_5 + (3h+6)\,\kappa_3\sigma^2\right]
\end{aligned}$$

For $s^{4,3}_{1,h}$, applying the indecomposable partition method for $\text{cum}\left(x^4_t, \widetilde{x}^3_{t-l}\right)$ yields

$$\begin{aligned}
&\left\langle x^4_t\widetilde{x}^3_{t-l}\right\rangle + 3\left\langle x^4_t\widetilde{x}_{t-l}\right\rangle\left\langle \widetilde{x}^2_{t-l}\right\rangle + 6\left\langle x^2_t\widetilde{x}^3_{t-l}\right\rangle\left\langle x^2_{t-l}\right\rangle + 12\left\langle x^3_t\widetilde{x}^2_{t-l}\right\rangle\left\langle x_t\widetilde{x}_{t-l}\right\rangle \\
&+4\left\langle x_t\widetilde{x}^3_{t-l}\right\rangle\left\langle x^3_t\right\rangle + 12\left\langle x^3_t\widetilde{x}_{t-l}\right\rangle\left\langle x_t\widetilde{x}^2_{t-l}\right\rangle + 18\left\langle x^2_t\widetilde{x}^2_{t-l}\right\rangle\left\langle x^2_t\widetilde{x}_{t-l}\right\rangle \\
&+12\left\langle x^3_t\right\rangle\left\langle \widetilde{x}^2_{t-l}\right\rangle\left\langle x_t\widetilde{x}_{t-l}\right\rangle + 18\left\langle x^2_t\widetilde{x}_{t-l}\right\rangle\left\langle x^2_t\right\rangle\left\langle \widetilde{x}^2_{t-l}\right\rangle \\
&+36\left\langle x_t\widetilde{x}^2_{t-l}\right\rangle\left\langle x^2_t\right\rangle\left\langle x_t\widetilde{x}_{t-l}\right\rangle + 36\left\langle x^2_t\widetilde{x}_{t-l}\right\rangle\left\langle x_t\widetilde{x}_{t-l}\right\rangle\left\langle x_t\widetilde{x}_{t-l}\right\rangle \\
= \;&\kappa_7 + (3h+18)\,\kappa_5\sigma^2 + 34\kappa_4\kappa_3 + (30h+72)\,\kappa_3\sigma^4.
\end{aligned}$$

Multiplying the above result by a factor of h gives rise to $s_{1,h}^{4,3}$. $s_{1,h}^{3,4}$ is a mirror image to $s_{1,h}^{4,3}$, so we have

$$\begin{aligned} s_{1,h}^{3,4} &= \sum \Big[\left\langle x_t^3 \widetilde{x}_{t-l}^4 \right\rangle + 6\left\langle x_t^3 \widetilde{x}_{t-l}^2 \right\rangle \left\langle \widetilde{x}_{t-l}^2 \right\rangle + 3\left\langle x_t \widetilde{x}_{t-l}^4 \right\rangle \left\langle x_t^2 \right\rangle + 12\left\langle x_t^2 \widetilde{x}_{t-l}^3 \right\rangle \left\langle x_t \widetilde{x}_{t-l} \right\rangle \\ &\quad +4\left\langle x_t^3 \widetilde{x}_{t-l} \right\rangle \left\langle \widetilde{x}_{t-l}^3 \right\rangle + 12\left\langle x_t \widetilde{x}_{t-l}^3 \right\rangle \left\langle x_t^2 \widetilde{x}_{t-l} \right\rangle + 18\left\langle x_t^2 \widetilde{x}_{t-l}^2 \right\rangle \left\langle x_t \widetilde{x}_{t-l}^2 \right\rangle \\ &\quad +12\left\langle \widetilde{x}_{t-l}^3 \right\rangle \left\langle x_t^2 \right\rangle \left\langle x_t \widetilde{x}_{t-l} \right\rangle + 18\left\langle x_t \widetilde{x}_{t-l}^2 \right\rangle \left\langle x_t^2 \right\rangle \left\langle \widetilde{x}_{t-l}^2 \right\rangle \\ &\quad +36\left\langle x_t^2 \widetilde{x}_{t-l} \right\rangle \left\langle \widetilde{x}_{t-l}^2 \right\rangle \left\langle x_t \widetilde{x}_{t-l} \right\rangle + 36\left\langle x_t \widetilde{x}_{t-l}^2 \right\rangle \left\langle x_t \widetilde{x}_{t-l} \right\rangle \left\langle x_t \widetilde{x}_{t-l} \right\rangle\Big] \\ &= h\left[\kappa_7 + (6h+15)\kappa_5\sigma^2 + (4h+30)\kappa_4\kappa_3 + (66h+36)\kappa_3\sigma^4\right] \end{aligned}$$

Replacing the x_t in the above with $\widetilde{x}_t$ yields the required $s_{h,h}^{3,4}$:

$$\begin{aligned} s_{h,h}^{3,4} &= \sum \Big[(h-|l|)\kappa_7 + \left(9h(h-|l|) + 12(h-|l|)^2\right)\kappa_5\sigma^2 \\ &\quad + \left(4h(h-|l|) + 30(h-|l|)^2\right)\kappa_4\kappa_3 \\ &\quad + \left(30h^2(h-|l|) + 36h(h-|l|)^2 + 36(h-|l|)^3\right)\kappa_3\sigma^4\Big] \\ &= h^2\kappa_7 + \left[9h^3 + 12A_h\right]\kappa_5\sigma^2 + \left[4h^3 + 30A_h\right]\kappa_4\kappa_3 \\ &\quad + \left[30h^4 + 36hA_h + 36B_h\right]\kappa_3\sigma^4, \end{aligned}$$

and this completes the proofs.

A.4 Estimation of cumulants

The covariance matrix S_0 is expressed in terms of cumulants, which in practice can be estimated using central moments as shown below.[11] Note that $\kappa_2 = \mu_2$ and $\kappa_3 = \mu_3$.

$$\kappa_4 = \mu_4 - 3\sigma^4, \quad (59)$$

$$\kappa_5 = \mu_5 - 10\mu_3\sigma^2, \quad (60)$$

$$\kappa_6 = \mu_6 - 15\mu_4\sigma^2 - 10\mu_3^2 + 30\sigma^6, \quad (61)$$

$$\kappa_7 = \mu_7 - 21\mu_5\sigma^2 - 35\mu_4\mu_3 + 210\mu_3\sigma^4, \quad (62)$$

$$\kappa_8 = \mu_8 - 28\mu_6\sigma^2 - 56\mu_5\mu_3 - 35\mu_4^2 + 420\mu_4\sigma^4 + 560\mu_3^2\sigma^2 - 630\sigma^8. \quad (63)$$

[11] See Kendall and Stuart (1969, p.71).

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Table 1. Empirical test sizes: all required moments exist

			Normal				Student with 9 d.f.			
N	h	a (%)	Skew	Kurt	Joint	McLi	Skew	Kurt	Joint	McLi
250	5	10	8.14	6.42	6.70	9.14	8.28	5.26	6.46	8.08
		5	4.32	3.16	4.16	4.96	4.30	2.92	4.02	4.88
		1	1.22	1.18	1.78	1.12	1.24	1.12	1.66	2.08
	10	10	7.06	4.26	5.76	9.08	7.08	4.02	5.74	8.76
		5	3.96	2.66	3.92	4.80	4.00	2.54	3.76	5.36
		1	1.28	1.18	1.62	1.48	1.10	1.12	1.80	1.84
	20	10	6.06	3.26	4.90	9.08	5.74	3.28	4.70	8.56
		5	3.04	2.38	3.12	5.04	3.38	2.08	2.88	5.46
		1	0.80	1.16	1.48	1.56	0.82	0.86	1.40	2.12
500	5	10	8.46	7.90	7.24	9.72	8.60	7.04	6.76	9.02
		5	4.42	3.92	4.14	5.20	4.20	3.48	4.24	5.50
		1	1.12	1.20	1.50	1.32	1.24	1.28	1.66	2.10
	10	10	7.74	6.14	6.56	9.92	8.26	5.76	6.88	9.00
		5	4.50	3.12	4.34	5.52	4.40	3.36	4.52	5.78
		1	1.44	1.46	2.00	1.64	1.54	1.50	2.12	2.14
	20	10	7.50	4.32	6.16	9.70	7.24	4.54	6.24	9.76
		5	3.88	3.18	3.98	5.16	4.40	3.00	4.44	6.06
		1	1.28	1.56	1.96	1.62	1.58	1.30	2.20	2.30
1000	5	10	8.92	9.18	7.64	10.46	8.48	8.36	8.12	9.04
		5	4.04	4.54	4.10	5.16	4.42	4.26	4.04	5.06
		1	0.92	0.86	1.12	1.32	0.86	1.24	1.36	1.88
	10	10	8.42	7.52	7.34	9.74	8.42	7.16	6.92	9.84
		5	4.62	3.68	4.40	5.32	4.30	3.44	4.26	6.18
		1	1.32	1.10	1.56	1.42	1.08	1.32	1.68	2.10
	20	10	8.56	6.02	7.12	9.74	8.50	5.98	6.82	10.46
		5	5.12	3.54	4.68	5.14	4.74	3.10	4.44	6.24
		1	1.48	1.30	2.00	1.54	1.48	1.22	2.12	2.08

5,000 replications are used to calculate the empirical test size. N, h and a are the number of observations, length of holding period and theoretical test size respectively. Skew, Kurt and Joint are respectively the skewness ratio test, kurtosis ratio test and their joint test. McLi is the squared-residual autocorrelations test of McLeod and Li (1983).

Table 2. Empirical test sizes: moment condition fails

N	h	a (%)	Student with 5 d.f. Skew	Kurt	Joint	McLi	Student with 3 d.f. Skew	Kurt	Joint	McLi
250	5	10	7.74	4.36	6.30	8.04	6.08	3.46	5.76	7.54
		5	4.24	2.74	3.98	5.44	3.72	2.36	3.86	5.52
		1	1.22	1.16	1.78	2.56	1.44	1.24	1.88	3.28
	10	10	6.52	3.46	5.52	8.66	5.74	3.08	5.36	8.60
		5	3.92	2.28	3.74	5.88	3.50	2.24	3.66	6.76
		1	1.16	1.20	1.92	2.60	1.40	1.08	1.94	3.52
	20	10	5.16	3.10	4.20	8.56	4.48	2.50	4.16	8.88
		5	3.04	2.04	2.86	5.66	2.78	1.86	3.06	6.48
		1	0.90	0.86	1.42	2.66	1.30	1.00	1.52	3.80
500	5	10	7.96	5.74	6.48	8.90	6.38	3.74	5.86	7.60
		5	4.06	2.88	4.44	6.18	3.38	2.28	4.06	5.94
		1	1.38	1.32	1.72	2.98	1.28	1.18	2.06	3.84
	10	10	7.52	4.82	6.34	9.12	6.06	3.40	5.76	8.64
		5	4.42	2.96	4.26	6.20	3.42	2.40	3.86	6.84
		1	1.46	1.54	2.20	3.16	1.30	1.40	2.10	4.50
	20	10	7.00	4.00	6.28	10.24	6.10	3.34	5.54	10.08
		5	4.22	2.78	4.16	7.06	3.38	2.16	3.72	8.08
		1	1.56	1.22	2.12	3.48	1.44	1.18	2.14	5.62
1000	5	10	8.42	6.90	7.28	8.32	7.10	4.12	5.94	7.00
		5	4.56	3.72	4.16	5.54	3.92	2.70	3.68	5.70
		1	0.94	1.28	1.72	2.96	1.14	1.42	1.80	3.58
	10	10	8.08	6.16	7.00	9.42	6.62	3.90	5.68	8.66
		5	4.70	3.48	4.20	6.78	3.76	2.56	3.78	6.92
		1	1.24	1.20	1.86	3.60	1.52	1.30	1.94	4.86
	20	10	7.98	5.20	6.72	10.70	6.84	4.14	6.28	10.02
		5	4.70	3.02	4.72	7.68	4.32	2.72	4.46	8.44
		1	1.66	1.12	2.04	4.00	1.74	1.18	2.20	5.80

5,000 replications are used to calculate the empirical test size. N, h and a are the number of observations, length of holding period and theoretical test size respectively. Skew, Kurt and Joint are respectively the skewness ratio test, kurtosis ratio test and their joint test. McLi is the squared-residual autocorrelations test of McLeod and Li (1983).

Table 3. Basic statistics

	No. of observations	*h*	Big Cap *sd*	Big Cap *sk*	Big Cap *ku*	Small Cap *sd*	Small Cap *sk*	Small Cap *ku*
US	2516	1	1.31	-0.33	10.30	1.67	-0.35	5.41
		5	1.16	-2.36	42.48	1.51	-1.77	31.23
		10	1.09	-4.75	91.14	1.42	-4.18	67.54
UK	2525	1	1.25	-0.14	7.55	0.79	-0.95	6.08
		5	1.16	-1.47	29.58	0.93	-2.00	26.59
		10	1.11	-3.28	44.26	1.01	-2.86	51.32
Germany	2539	1	1.45	0.03	5.81	1.17	-0.55	5.69
		5	1.41	-1.70	19.97	1.31	-2.75	30.46
		10	1.37	-3.70	40.89	1.31	-4.97	61.46
Japan	2540	1	1.61	-0.51	7.71	1.26	-0.97	11.73
		5	1.52	-1.56	33.47	1.34	-2.66	42.15
		10	1.48	-2.86	44.89	1.31	-3.51	48.03

sd, sk and ku are the standard deviation, skewness, excessive kurtosis respectively. Note that all statistics are scaled so that if the returns are IID, the figures should remain unchange with regard to h.

Table 4. Tests on Raw Returns

	h	LB	McLi	Skew	Kurt	Joint	*sd*	*k3*	*k4*
				Big Capitalization Stock Index					
	1						1.31	-0.32	10.27
US	5	107.4	3987.5	14.0	5.1	17.7	1.16	-1.63	26.00
	10	126.2	5201.6	16.2	4.2	19.5	1.09	-2.75	44.08
	1						1.25	-0.14	7.52
UK	5	51.5	2893.7	10.9	7.6	18.1	1.16	-1.16	21.57
	10	77.2	3631.1	15.6	0.9	16.6	1.11	-2.26	26.80
	1						1.45	0.03	5.79
Germany	5	35.9	1928.1	32.5	10.3	53.7	1.41	-1.56	17.77
	10	53.7	2429.9	41.8	8.0	57.8	1.37	-3.13	32.58
	1						1.61	-0.51	7.69
Japan	5	18.1	2979.4	6.7	17.4	19.3	1.52	-1.33	26.82
	10	41.6	3188.6	10.3	4.0	11.7	1.48	-2.26	32.36
				Small Capitalization Stock Index					
	1						1.67	-0.34	5.39
US	5	78.9	4128.5	11.4	11.3	17.2	1.50	-1.29	20.71
	10	93.0	5420.0	19.3	5.1	21.5	1.42	-2.57	35.42
	1						0.79	-0.94	6.06
UK	5	159.2	2054.5	52.9	283.4	329.2	0.93	-3.27	51.18
	10	172.5	2328.0	60.2	536.3	614.4	1.01	-5.94	135.83
	1						1.17	-0.55	5.66
Germany	5	85.7	2031.3	131.7	237.5	289.5	1.31	-3.91	48.54
	10	92.3	2224.5	158.3	194.3	261.2	1.31	-7.07	98.00
	1						1.26	-0.97	11.69
Japan	5	41.8	1002.5	34.2	69.4	71.7	1.34	-3.17	53.24
	10	59.2	1008.3	23.0	21.6	29.3	1.31	-3.96	56.07

Diagnostic tests are carried out on GARCH-filtered returns. *h* refers to the length of holding period. LB is the Ljung-Box test whereas McLi, Skew, Kurt and Joint refer to the same tests as in Table 1 and 2. *sd* , *k3* and *k4* are respectively the scaled standard deviation, standardized third and fourth cumulants. There are about 2,500 observations in each time series and the reported test statistics are Chi square test statitics. Dark (light) shade indicates significance at 1% (5%) level.

Table 5. Tests on AR(10)-filtered Returns

	h	LB	McLi	Skew	Kurt	Joint	*sd*	*k3*	*k4*
				Big Capitalization Stock Index					
	1						1.30	-0.58	9.32
US	5	46.2	4147.7	40.6	61.7	77.8	1.30	-2.71	42.17
	10	59.8	5345.5	55.7	72.3	98.6	1.29	-4.78	85.50
	1						1.24	-0.32	6.73
UK	5	17.6	2740.3	18.6	38.3	48.0	1.24	-1.59	28.15
	10	37.9	3420.0	33.5	18.0	44.1	1.23	-3.25	40.47
	1						1.44	-0.05	5.33
Germany	5	21.0	2038.3	47.0	24.8	81.6	1.44	-1.92	20.66
	10	37.8	2533.8	64.5	26.3	95.5	1.46	-3.87	41.46
	1						1.61	-0.64	7.63
Japan	5	8.3	3039.3	12.1	39.4	40.0	1.61	-1.74	32.15
	10	30.2	3246.2	18.6	20.3	27.7	1.62	-2.97	44.31
				Small Capitalization Stock Index					
	1						1.65	-0.53	5.25
US	5	38.9	4083.2	38.7	70.8	76.9	1.65	-2.24	31.34
	10	50.5	5350.7	63.6	73.1	97.8	1.66	-4.36	66.71
	1						0.78	-0.67	5.59
UK	5	42.6	1959.1	14.0	42.0	42.0	0.78	-1.73	26.52
	10	55.6	2231.6	20.9	46.3	48.4	0.78	-2.92	54.06
	1						1.15	-0.43	6.13
Germany	5	26.0	1757.4	77.5	59.1	119.0	1.15	-2.96	32.83
	10	34.3	1921.9	93.1	59.3	129.1	1.17	-5.21	65.24
	1						1.26	-0.76	10.95
Japan	5	11.6	1094.7	31.7	37.6	49.6	1.26	-2.80	42.13
	10	30.5	1104.1	20.3	12.0	23.9	1.25	-3.48	46.26

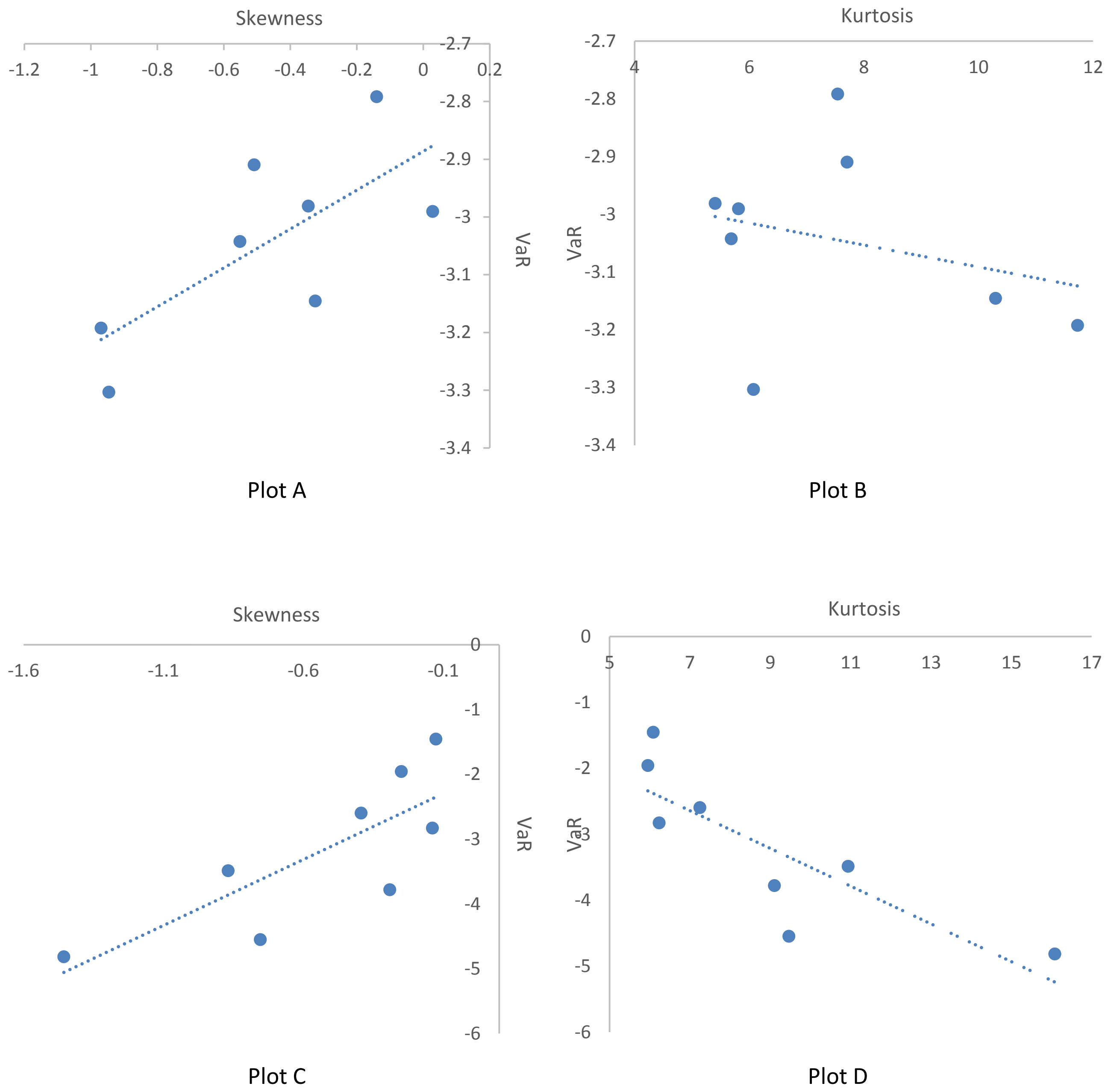
Diagnostic tests are carried out on GARCH-filtered returns. *h* refers to the length of holding period. LB is the Ljung-Box test whereas McLi, Skew, Kurt and Joint refer to the same tests as in Table 1 and 2. *sd* , *k3* and *k4* are respectively the scaled standard deviation, standardized third and fourth cumulants. There are about 2,500 observations in each time series and the reported test statistics are Chi square test statitics. Dark (light) shade indicates significance at 1% (5%) level.

Table 6. Tests on returns filtered by AR(1)-GARCH-Normal

	h	LB	McLi	Skew	Kurt	Joint	*sd*	*k3*	*k4*
Big Capitalization Stock Index									
US	1						1.00	-0.52	1.45
	5	26.73	26.15	11.54	0.06	15.59	0.98	-1.23	3.15
	10	33.28	33.87	5.23	0.88	9.39	0.93	-1.44	4.22
UK	1						1.00	-0.34	0.77
	5	11.19	23.51	18.08	1.98	18.14	1.01	-1.15	2.75
	10	21.42	37.19	14.03	0.27	14.36	0.99	-1.72	4.46
Germany	1						1.00	-0.36	1.21
	5	13.79	22.45	29.15	4.59	29.24	1.01	-1.43	4.39
	10	21.15	25.74	30.25	4.95	30.45	1.00	-2.45	12.67
Japan	1						1.00	-0.42	0.85
	5	10.57	26.92	16.74	9.03	17.88	1.04	-1.22	4.11
	10	15.69	37.51	8.69	2.20	8.81	1.04	-1.54	3.68
Small Capitalization Stock Index									
US	1						1.00	-0.36	0.59
	5	20.93	30.96	12.71	0.07	16.42	0.98	-1.03	1.51
	10	25.44	38.41	7.08	0.49	9.34	0.93	-1.35	4.34
UK	1						1.00	-0.65	2.00
	5	36.44	13.87	14.26	14.76	17.50	1.07	-1.51	5.48
	10	42.52	24.68	3.45	13.84	13.85	1.13	-1.47	2.94
Germany	1						1.00	-0.61	2.34
	5	18.63	9.89	18.84	4.86	19.76	1.04	-1.60	5.31
	10	30.23	13.65	15.16	2.84	15.46	1.04	-2.23	7.56
Japan	1						1.00	-0.81	3.01
	5	19.26	12.27	12.96	6.14	13.32	1.06	-1.71	6.48
	10	25.58	16.77	5.53	4.75	6.21	1.09	-1.87	4.13

Diagnostic tests are carried out on GARCH-filtered returns. *h* refers to the length of holding period. LB is the Ljung-Box test whereas McLi, Skew, Kurt and Joint refer to the same tests as in Table 1 and 2. *sd* , *k3* and *k4* are respectively the scaled standard deviation, standardized third and fourth cumulants. There are about 2,500 observations in each time series and the reported test statistics are Chi square test statitics. Dark (light) shade indicates significance at 1% (5%) level.

Figure 1. A scatter plot of standardized VaR against skewness and kurtosis



Scatter plot A and B are based on 10 years of data from 2006 to 2015 whereas scatter plot C and D are based on 30 years of data from 1986 to 2015. The skewness, kurtosis and VaR are measures of single-period returns with the latter obtained by dividing the first percentile of the returns by the associated standard deviation.